Exercises 4.8

Finding Antiderivatives

In Exercises 1–24, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$ b. x^2 c. $x^2 - 2x + 1$
2. a. $6x$ b. x^7 c. $x^7 - 6x + 8$
3. a. $-3x^{-4}$ b. x^{-4} c. $x^{-4} + 2x + 3$
4. a. $2x^{-3}$ b. $\frac{x^{-3}}{2} + x^2$ c. $-x^{-3} + x - 1$
5. a. $\frac{1}{x^2}$ b. $\frac{5}{x^2}$ c. $2 - \frac{5}{x^2}$
6. a. $-\frac{2}{x^3}$ b. $\frac{1}{2x^3}$ c. $x^3 - \frac{1}{x^3}$
7. a. $\frac{3}{2}\sqrt{x}$ b. $\frac{1}{2\sqrt{x}}$ c. $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. a. $\frac{4}{3}\sqrt[3]{x}$ b. $\frac{1}{3\sqrt[3]{x}}$ c. $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
9. a. $\frac{2}{3}x^{-1/3}$ b. $\frac{1}{3}x^{-2/3}$ c. $-\frac{1}{3}x^{-4/3}$
10. a. $\frac{1}{2}x^{-1/2}$ b. $-\frac{1}{2}x^{-3/2}$ c. $-\frac{3}{2}x^{-5/2}$
11. a. $\frac{1}{x}$ b. $\frac{7}{x}$ c. $1 - \frac{5}{x}$
12. a. $\frac{1}{3x}$ b. $\frac{2}{5x}$ c. $1 + \frac{4}{3x} - \frac{1}{x^2}$
13. a. $-\pi \sin \pi x$ b. $3 \sin x$ c. $\sin \pi x - 3 \sin 3x$
14. a. $\pi \cos \pi x$ b. $\frac{\pi}{2} \cos \frac{\pi x}{2}$ c. $\cos \frac{\pi x}{2} + \pi \cos x$
15. a. $\sec^2 x$ b. $\frac{2}{3} \sec^2 \frac{x}{3}$ c. $-\sec^2 \frac{3x}{2}$
16. a. $\csc^2 x$ b. $-\frac{3}{2} \csc^2 \frac{3x}{2}$ c. $1 - 8 \csc^2 2x$
17. a. $\csc x \cot x$ b. $-\csc 5x \cot 5x$ c. $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
18. a. $\sec x \tan x$ b. $4 \sec 3x \tan 3x$ c. $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$
19. a. e^{3x} b. e^{-x} c. $e^{x/2}$
20. a. e^{-2x} b. $e^{4x/3}$ c. $e^{-x/5}$
21. a. 3^x b. 2^{-x} c. $\left(\frac{5}{3}\right)^x$
22. a. $x^{\sqrt{3}}$ b. x^{π} c. $x^{\sqrt{2}-1}$
23. a. $\frac{2}{\sqrt{1 - x^2}}$ b. $\frac{1}{2(x^2 + 1)}$ c. $\frac{1}{1 + 4x^2}$
24. a. $x - \left(\frac{1}{2}\right)^x$ b. $x^2 + 2^x$ c. $\pi^x - x^{-1}$

Finding Indefinite Integrals

In Exercises 25–70, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

25. $\int (x + 1)\, dx$
26. $\int (5 - 6x)\, dx$
27. $\int \left(3t^2 + \frac{t}{2}\right) dt$
28. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$
29. $\int (2x^3 - 5x + 7)\, dx$
30. $\int (1 - x^2 - 3x^5)\, dx$
31. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$
32. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
33. $\int x^{-1/3}\, dx$
34. $\int x^{-5/4}\, dx$
35. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$
36. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
37. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$
38. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$
39. $\int 2x(1 - x^{-3})\, dx$
40. $\int x^{-3}(x + 1)\, dx$
41. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$
42. $\int \frac{4 + \sqrt{t}}{t^3}\, dt$
43. $\int (-2 \cos t)\, dt$
44. $\int (-5 \sin t)\, dt$
45. $\int 7 \sin \frac{\theta}{3}\, d\theta$
46. $\int 3 \cos 5\theta\, d\theta$
47. $\int (-3 \csc^2 x)\, dx$
48. $\int \left(-\frac{\sec^2 x}{3}\right) dx$
49. $\int \frac{\csc \theta \cot \theta}{2}\, d\theta$
50. $\int \frac{2}{5} \sec \theta \tan \theta\, d\theta$
51. $\int (e^{3x} + 5e^{-x})\, dx$
52. $\int (2e^x - 3e^{-2x})\, dx$
53. $\int (e^{-x} + 4^x)\, dx$
54. $\int (1.3)^x\, dx$
55. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$
56. $\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$
57. $\int (\sin 2x - \csc^2 x)\, dx$
58. $\int (2 \cos 2x - 3 \sin 3x)\, dx$
59. $\int \frac{1 + \cos 4t}{2}\, dt$
60. $\int \frac{1 - \cos 6t}{2}\, dt$
61. $\int \left(\frac{1}{x} - \frac{5}{x^2 + 1}\right) dx$
62. $\int \left(\frac{2}{\sqrt{1 - y^2}} - \frac{1}{y^{1/4}}\right) dy$
63. $\int 3x^{\sqrt{3}}\, dx$
64. $\int x^{\sqrt{2}-1}\, dx$

65. $\int (1 + \tan^2 \theta)\, d\theta$

66. $\int (2 + \tan^2 \theta)\, d\theta$

(*Hint:* $1 + \tan^2 \theta = \sec^2 \theta$)

67. $\int \cot^2 x\, dx$

68. $\int (1 - \cot^2 x)\, dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

69. $\int \cos \theta (\tan \theta + \sec \theta)\, d\theta$

70. $\int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta$

Checking Antiderivative Formulas

Verify the formulas in Exercises 71–82 by differentiation.

71. $\int (7x - 2)^3\, dx = \frac{(7x - 2)^4}{28} + C$

72. $\int (3x + 5)^{-2}\, dx = -\frac{(3x + 5)^{-1}}{3} + C$

73. $\int \sec^2 (5x - 1)\, dx = \frac{1}{5} \tan (5x - 1) + C$

74. $\int \csc^2\left(\frac{x - 1}{3}\right) dx = -3 \cot\left(\frac{x - 1}{3}\right) + C$

75. $\int \frac{1}{(x + 1)^2} dx = -\frac{1}{x + 1} + C$

76. $\int \frac{1}{(x + 1)^2} dx = \frac{x}{x + 1} + C$

77. $\int \frac{1}{x + 1}\, dx = \ln |x + 1| + C, \quad x \neq -1$

78. $\int xe^x\, dx = xe^x - e^x + C$

79. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

80. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

81. $\int \frac{\tan^{-1} x}{x^2}\, dx = \ln x - \frac{1}{2} \ln (1 + x^2) - \frac{\tan^{-1} x}{x} + C$

82. $\int (\sin^{-1} x)^2\, dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2} \sin^{-1} x + C$

83. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int x \sin x\, dx = \frac{x^2}{2} \sin x + C$

b. $\int x \sin x\, dx = -x \cos x + C$

c. $\int x \sin x\, dx = -x \cos x + \sin x + C$

84. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{\sec^3 \theta}{3} + C$

b. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \tan^2 \theta + C$

c. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \sec^2 \theta + C$

85. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int (2x + 1)^2\, dx = \frac{(2x + 1)^3}{3} + C$

b. $\int 3(2x + 1)^2\, dx = (2x + 1)^3 + C$

c. $\int 6(2x + 1)^2\, dx = (2x + 1)^3 + C$

86. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x + C}$

b. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x} + C$

c. $\int \sqrt{2x + 1}\, dx = \frac{1}{3}\left(\sqrt{2x + 1}\right)^3 + C$

87. Right, or wrong? Give a brief reason why.

$$\int \frac{-15(x + 3)^2}{(x - 2)^4} dx = \left(\frac{x + 3}{x - 2}\right)^3 + C$$

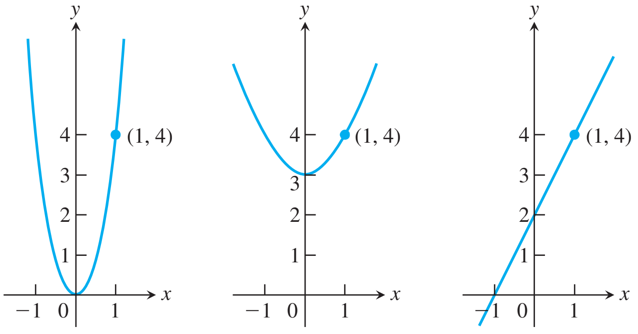
88. Right, or wrong? Give a brief reason why.

$$\int \frac{x \cos (x^2) - \sin (x^2)}{x^2}\, dx = \frac{\sin (x^2)}{x} + C$$

Initial Value Problems

89. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$

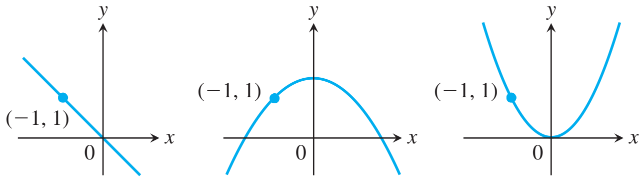


(a) (b) (c)

Give reasons for your answer.

90. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



(a) (b) (c)

Give reasons for your answer.

Solve the initial value problems in Exercises 91–112.

91. $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

92. $\frac{dy}{dx} = 10 - x, \quad y(0) = -1$

93. $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

94. $\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

95. $\frac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

96. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad y(4) = 0$

97. $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

98. $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

99. $\frac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

100. $\frac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

101. $\frac{dv}{dt} = \frac{1}{2}\sec t \tan t, \quad v(0) = 1$

102. $\frac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\frac{\pi}{2}\right) = -7$

103. $\frac{dv}{dt} = \frac{3}{t\sqrt{t^2 - 1}}, \quad t > 1, v(2) = 0$

104. $\frac{dv}{dt} = \frac{8}{1 + t^2} + \sec^2 t, \quad v(0) = 1$

105. $\frac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

106. $\frac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

107. $\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left.\frac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

108. $\frac{d^2s}{dt^2} = \frac{3t}{8}; \quad \left.\frac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

109. $\frac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

110. $\frac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\frac{1}{2}, \quad \theta(0) = \sqrt{2}$

111. $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

112. $y^{(4)} = -\cos x + 8 \sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

113. Find the curve $y = f(x)$ in the xy-plane that passes through the point $(9, 4)$ and whose slope at each point is $3\sqrt{x}$.

114. a. Find a curve $y = f(x)$ with the following properties:

i) $\frac{d^2y}{dx^2} = 6x$

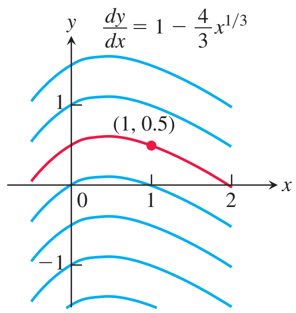
ii) Its graph passes through the point $(0, 1)$ and has a horizontal tangent there.

b. How many curves like this are there? How do you know?

Solution (Integral) Curves

Exercises 115–118 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

115.



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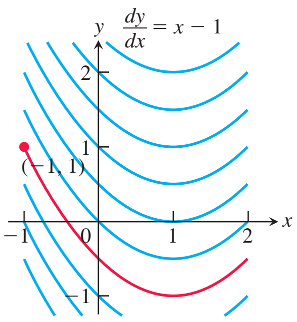

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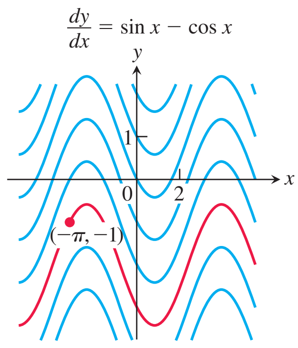

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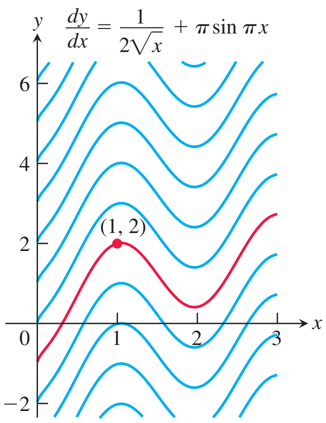

Applications

119. Finding displacement from an antiderivative of velocity

a. Suppose that the velocity of a body moving along the s-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

i) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

ii) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

iii) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

b. Suppose that the position s of a body moving along a coordinate line is a differentiable function of time t. Is it true that once you know an antiderivative of the velocity function ds/dt you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

120. Liftoff from Earth A rocket lifts off the surface of Earth with a constant acceleration of $20\ \text{m/sec}^2$. How fast will the rocket be going 1 min later?

121. Stopping a car in time You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

1. Solve the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = -k \qquad (k \text{ constant})$

Initial conditions: $\dfrac{ds}{dt} = 88$ and $s = 0$ when $t = 0$.

Measuring time and distance from when the brakes are applied

2. Find the value of t that makes $ds/dt = 0$. (The answer will involve k.)

3. Find the value of k that makes $s = 242$ for the value of t you found in Step 2.

122. Stopping a motorcycle The State of Illinois Cycle Rider Safety Program requires motorcycle riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

123. Motion along a coordinate line A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - \left(3/\sqrt{t}\right)$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

a. the velocity $v = ds/dt$ in terms of t

b. the position s in terms of t.

T **124. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for s as a function of t. Then find the value of t that makes s equal to 0.

Differential equation: $\dfrac{d^2s}{dt^2} = -5.2\ \text{ft/sec}^2$

Initial conditions: $\dfrac{ds}{dt} = 0$ and $s = 4$ when $t = 0$

125. Motion with constant acceleration The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \qquad (1)$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\dfrac{d^2s}{dt^2} = a$

Initial conditions: $\dfrac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$.

126. Free fall near the surface of a planet For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of g length-units/sec^2, Equation (1) in Exercise 125 takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \qquad (2)$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s. The velocity v_0 is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 125, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

127. Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

a. $\int f(x)\,dx$ **b.** $\int g(x)\,dx$

c. $\int [-f(x)]\,dx$ **d.** $\int [-g(x)]\,dx$

e. $\int [f(x) + g(x)]\,dx$ **f.** $\int [f(x) - g(x)]\,dx$

128. Uniqueness of solutions If differentiable functions $y = F(x)$ and $y = g(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0,$$

on an interval I, must $F(x) = G(x)$ for every x in I? Give reasons for your answer.

COMPUTER EXPLORATIONS

Use a CAS to solve the initial value problems in Exercises 129–132. Plot the solution curves.

129. $y' = \cos^2 x + \sin x, \quad y(\pi) = 1$

130. $y' = \dfrac{1}{x} + x, \quad y(1) = -1$

131. $y' = \dfrac{1}{\sqrt{4 - x^2}}, \quad y(0) = 2$

132. $y'' = \dfrac{2}{x} + \sqrt{x}, \quad y(1) = 0, \quad y'(1) = 0$