The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function $f$ and the $x$-axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.10. We will see in the next section that if the function $f$ is continuous over the closed interval $[a, b]$, then no matter how we choose the partition $P$ and the points $c_k$ in its subintervals to construct a Riemann sum, a single limiting value is approached as the subinterval widths, controlled by the norm of the partition, approach zero.

## Exercises 5.2

### Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

**1.** $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$ **2.** $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

**3.** $\displaystyle\sum_{k=1}^{4} \cos k\pi$ **4.** $\displaystyle\sum_{k=1}^{5} \sin k\pi$

**5.** $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$ **6.** $\displaystyle\sum_{k=1}^{4} (-1)^{k} \cos k\pi$

**7.** Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

**a.** $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} 2^{k}$ **c.** $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

**8.** Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

**a.** $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} (-1)^{k} 2^{k}$ **c.** $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

**9.** Which formula is not equivalent to the other two?

**a.** $\displaystyle\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\displaystyle\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ **c.** $\displaystyle\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

**10.** Which formula is not equivalent to the other two?

**a.** $\displaystyle\sum_{k=1}^{4} (k-1)^2$ **b.** $\displaystyle\sum_{k=-1}^{3} (k+1)^2$ **c.** $\displaystyle\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

**11.** $1 + 2 + 3 + 4 + 5 + 6$ **12.** $1 + 4 + 9 + 16$

**13.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ **14.** $2 + 4 + 6 + 8 + 10$

**15.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ **16.** $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

### Values of Finite Sums

**17.** Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

**a.** $\displaystyle\sum_{k=1}^{n} 3a_k$ **b.** $\displaystyle\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\displaystyle\sum_{k=1}^{n} (a_k + b_k)$

**d.** $\displaystyle\sum_{k=1}^{n} (a_k - b_k)$ **e.** $\displaystyle\sum_{k=1}^{n} (b_k - 2a_k)$

**18.** Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

**a.** $\displaystyle\sum_{k=1}^{n} 8a_k$ **b.** $\displaystyle\sum_{k=1}^{n} 250b_k$

**c.** $\displaystyle\sum_{k=1}^{n} (a_k + 1)$ **d.** $\displaystyle\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

**19.** **a.** $\displaystyle\sum_{k=1}^{10} k$ **b.** $\displaystyle\sum_{k=1}^{10} k^2$ **c.** $\displaystyle\sum_{k=1}^{10} k^3$

**20.** **a.** $\displaystyle\sum_{k=1}^{13} k$ **b.** $\displaystyle\sum_{k=1}^{13} k^2$ **c.** $\displaystyle\sum_{k=1}^{13} k^3$

**21.** $\displaystyle\sum_{k=1}^{7} (-2k)$ **22.** $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$

**23.** $\displaystyle\sum_{k=1}^{6} (3 - k^2)$ **24.** $\displaystyle\sum_{k=1}^{6} (k^2 - 5)$

**25.** $\displaystyle\sum_{k=1}^{5} k(3k + 5)$ **26.** $\displaystyle\sum_{k=1}^{7} k(2k + 1)$

**27.** $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ **28.** $\displaystyle\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

**29.** **a.** $\displaystyle\sum_{k=1}^{7} 3$ **b.** $\displaystyle\sum_{k=1}^{500} 7$ **c.** $\displaystyle\sum_{k=3}^{264} 10$

**30.** **a.** $\displaystyle\sum_{k=9}^{36} k$ **b.** $\displaystyle\sum_{k=3}^{17} k^2$ **c.** $\displaystyle\sum_{k=18}^{71} k(k-1)$

**31.** **a.** $\displaystyle\sum_{k=1}^{n} 4$ **b.** $\displaystyle\sum_{k=1}^{n} c$ **c.** $\displaystyle\sum_{k=1}^{n} (k-1)$

**32.** **a.** $\displaystyle\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ **b.** $\displaystyle\sum_{k=1}^{n} \frac{c}{n}$ **c.** $\displaystyle\sum_{k=1}^{n} \frac{k}{n^2}$

### Riemann Sums

In Exercises 33–36, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that $c_k$ is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the $k$th subinterval. (Make a separate sketch for each set of rectangles.)

**33.** $f(x) = x^2 - 1, \quad [0, 2]$

**34.** $f(x) = -x^2, \quad [0, 1]$

**35.** $f(x) = \sin x, \quad [-\pi, \pi]$

**36.** $f(x) = \sin x + 1, \quad [-\pi, \pi]$

**37.** Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

**38.** Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

### Limits of Riemann Sums

For the functions in Exercises 39–46, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into $n$ equal subintervals and using the right-hand endpoint for each $c_k$. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

**39.** $f(x) = 1 - x^2$ over the interval $[0, 1]$.

**40.** $f(x) = 2x$ over the interval $[0, 3]$.

**41.** $f(x) = x^2 + 1$ over the interval $[0, 3]$.

**42.** $f(x) = 3x^2$ over the interval $[0, 1]$.

**43.** $f(x) = x + x^2$ over the interval $[0, 1]$.

**44.** $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.

**45.** $f(x) = 2x^3$ over the interval $[0, 1]$.

**46.** $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.

# 5.3 The Definite Integral

In Section 5.2 we investigated the limit of a finite sum for a function defined over a closed interval $[a, b]$ using $n$ subintervals of equal width (or length), $(b - a)/n$. In this section we consider the limit of more general Riemann sums as the norm of the partitions of $[a, b]$ approaches zero. For general Riemann sums, the subintervals of the partitions need not have equal widths. The limiting process then leads to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

## Definition of the Definite Integral

The definition of the definite integral is based on the idea that for certain functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann sums approach a limiting value $J$. What we mean by this limit is that a Riemann sum will be close to the number $J$ provided that the norm of its partition is sufficiently small (so that all of its subintervals have thin enough widths). We introduce the symbol $\epsilon$ as a small positive number that specifies how close to $J$ the Riemann sum must be, and the symbol $\delta$ as a second small positive number that specifies how small the norm of a partition must be in order for convergence to happen. We now define this limit precisely.

**DEFINITION** Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number $J$ is the **definite integral of $f$ over $[a, b]$** and that $J$ is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of $c_k$ in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$

The definition involves a limiting process in which the norm of the partition goes to zero.

We have many choices for a partition $P$ with norm going to zero, and many choices of points $c_k$ for each partition. The definite integral exists when we always get the same limit $J$, no matter what choices are made. When the limit exists we write it as the definite integral

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$