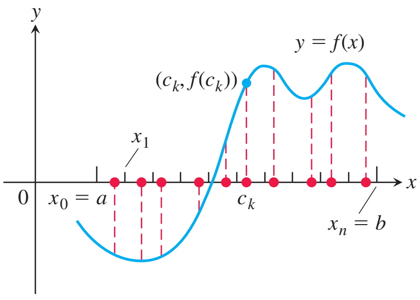


FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

Alternatively, we can use the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n. A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way. We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ and evaluate f at a point c_k in each (Figure 5.14). The average of the n sampled values is

$$\begin{aligned}\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n}\sum_{k=1}^{n} f(c_k) \\ &= \frac{\Delta x}{b - a}\sum_{k=1}^{n} f(c_k) \qquad \Delta x = \frac{b-a}{n}, \text{ so } \frac{1}{n} = \frac{\Delta x}{b-a} \\ &= \frac{1}{b - a}\sum_{k=1}^{n} f(c_k)\,\Delta x. \qquad \text{Constant Multiple Rule}\end{aligned}$$

The average is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the size of the sample and let the norm of the partition approach zero, the average approaches $(1/(b - a))\int_a^b f(x)\,dx$. Both points of view lead us to the following definition.

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

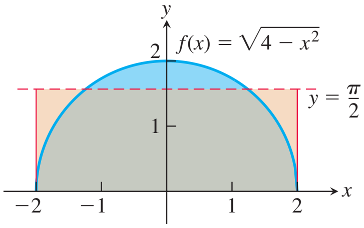


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

EXAMPLE 5 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is the upper semicircle of radius 2 centered at the origin (Figure 5.15).

Since we know the area inside a circle, we do not need to take the limit of Riemann sums. The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi(2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2} \sqrt{4 - x^2}\,dx = 2\pi.$$

Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2 - (-2)}\int_{-2}^{2} \sqrt{4 - x^2}\,dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Notice that the average value of f over $[-2, 2]$ is the same as the height of a rectangle over $[-2, 2]$ whose area equals the area of the upper semicircle (see Figure 5.15). ■

Exercises 5.3

Interpreting Limits of Sums as Integrals

Express the limits in Exercises 1–8 as definite integrals.

1. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k^{\,2}\,\Delta x_k$, where P is a partition of $[0, 2]$

2. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} 2c_k^{\,3}\,\Delta x_k$, where P is a partition of $[-1, 0]$

3. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (c_k^{\,2} - 3c_k)\,\Delta x_k$, where P is a partition of $[-7, 5]$

4. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right)\Delta x_k$, where P is a partition of $[1, 4]$

5. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k}\,\Delta x_k$, where P is a partition of $[2, 3]$

6. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\, \Delta x_k$, where P is a partition of $[0, 1]$

7. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sec c_k)\, \Delta x_k$, where P is a partition of $[-\pi/4, 0]$

8. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\tan c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/4]$

Using the Definite Integral Rules

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.6 to find

a. $\int_2^2 g(x)\,dx$ **b.** $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ **d.** $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ **f.** $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.6 to find

a. $\int_1^9 -2f(x)\,dx$ **b.** $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ **d.** $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ **f.** $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ **b.** $\int_1^2 \sqrt{3} f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ **d.** $\int_1^2 [-f(x)]\,dx$

12. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\int_0^{-3} g(t)\,dt$ **b.** $\int_{-3}^0 g(u)\,du$

c. $\int_{-3}^0 [-g(x)]\,dx$ **d.** $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

13. Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\int_3^4 f(z)\,dz$ **b.** $\int_4^3 f(t)\,dt$

14. Suppose that h is integrable and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a. $\int_1^3 h(r)\,dr$ **b.** $-\int_3^1 h(u)\,du$

Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use known area formulas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$ **16.** $\int_{1/2}^{3/2} (-2x + 4)\,dx$

17. $\int_{-3}^3 \sqrt{9 - x^2}\,dx$ **18.** $\int_{-4}^0 \sqrt{16 - x^2}\,dx$

19. $\int_{-2}^1 |x|\,dx$ **20.** $\int_{-1}^1 (1 - |x|)\,dx$

21. $\int_{-1}^1 (2 - |x|)\,dx$ **22.** $\int_{-1}^1 \left(1 + \sqrt{1 - x^2}\right) dx$

Use known area formulas to evaluate the integrals in Exercises 23–28.

23. $\int_0^b \frac{x}{2}\,dx, \quad b > 0$ **24.** $\int_0^b 4x\,dx, \quad b > 0$

25. $\int_a^b 2s\,ds, \quad 0 < a < b$ **26.** $\int_a^b 3t\,dt, \quad 0 < a < b$

27. $f(x) = \sqrt{4 - x^2}$ on **a.** $[-2, 2]$, **b.** $[0, 2]$

28. $f(x) = 3x + \sqrt{1 - x^2}$ on **a.** $[-1, 0]$, **b.** $[-1, 1]$

Evaluating Definite Integrals

Use the results of Equations (2) and (4) to evaluate the integrals in Exercises 29–40.

29. $\int_1^{\sqrt{2}} x\,dx$ **30.** $\int_{0.5}^{2.5} x\,dx$ **31.** $\int_\pi^{2\pi} \theta\,d\theta$

32. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$ **33.** $\int_0^{\sqrt[3]{7}} x^2\,dx$ **34.** $\int_0^{0.3} s^2\,ds$

35. $\int_0^{1/2} t^2\,dt$ **36.** $\int_0^{\pi/2} \theta^2\,d\theta$ **37.** $\int_a^{2a} x\,dx$

38. $\int_a^{\sqrt{3}a} x\,dx$ **39.** $\int_0^{\sqrt[3]{b}} x^2\,dx$ **40.** $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.6 and Equations (2)–(4) to evaluate the integrals in Exercises 41–50.

41. $\int_3^1 7\,dx$ **42.** $\int_0^2 5x\,dx$

43. $\int_0^2 (2t - 3)\,dt$ **44.** $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt$

45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz$ **46.** $\int_3^0 (2z - 3)\,dz$

47. $\int_1^2 3u^2\,du$ **48.** $\int_{1/2}^1 24u^2\,du$

49. $\int_0^2 (3x^2 + x - 5)\,dx$ **50.** $\int_1^0 (3x^2 + x - 5)\,dx$

Finding Area by Definite Integrals

In Exercises 51–54, use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

51. $y = 3x^2$ **52.** $y = \pi x^2$

53. $y = 2x$ **54.** $y = \frac{x}{2} + 1$

Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$

56. $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$

57. $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t - 1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

Definite Integrals as Limits of Sums

Use the method of Example 4a or Equation (1) to evaluate the definite integrals in Exercises 63–70.

63. $\displaystyle\int_a^b c\,dx$

64. $\displaystyle\int_0^2 (2x + 1)\,dx$

65. $\displaystyle\int_a^b x^2\,dx,\quad a < b$

66. $\displaystyle\int_{-1}^0 (x - x^2)\,dx$

67. $\displaystyle\int_{-1}^2 (3x^2 - 2x + 1)\,dx$

68. $\displaystyle\int_{-1}^1 x^3\,dx$

69. $\displaystyle\int_a^b x^3\,dx,\quad a < b$

70. $\displaystyle\int_0^1 (3x - x^3)\,dx$

Theory and Examples

71. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\,dx?$$

(*Hint:* Where is the integrand positive?)

72. What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\,dx?$$

73. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

74. (*Continuation of Exercise 73.*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}\,dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1 + x^2}\,dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

75. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

76. Show that the value of $\int_0^1 \sqrt{x + 8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

77. Integrals of nonnegative functions Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \ge 0.$$

78. Integrals of nonpositive functions Show that if f is integrable then

$$f(x) \le 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \le 0.$$

79. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

80. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

81. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the constant function $\text{av}(f)$ should have the same integral over $[a, b]$ as f. Does it? That is, does

$$\int_a^b \text{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

82. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

a. $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$

b. $\text{av}(kf) = k\,\text{av}(f)$ (any number k)

c. $\text{av}(f) \le \text{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

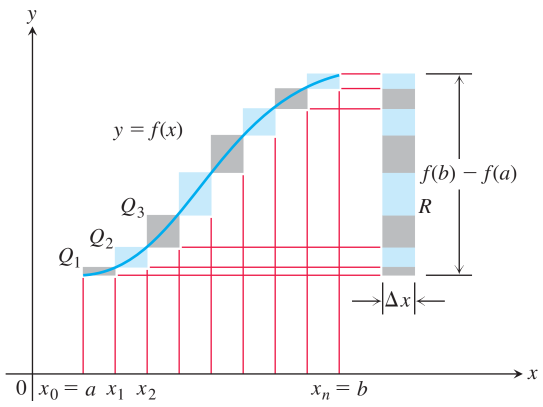
83. Upper and lower sums for increasing functions

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie approximately along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\,\Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.



84. Upper and lower sums for decreasing functions (*Continuation of Exercise 83.*)

a. Draw a figure like the one in Exercise 83 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 83a.

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \le |f(b) - f(a)|\ \Delta x_{\max}$$

of Exercise 83b still holds and hence that $\lim_{\|P\|\to 0} (U - L) = 0$.

85. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b. Find the limit of U as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

86. Suppose that f is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}$, which need not be equal.

a. If $m_k = \min\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **lower sum**

$$L = m_1\,\Delta x_1 + m_2\,\Delta x_2 + \cdots + m_n\,\Delta x_n$$

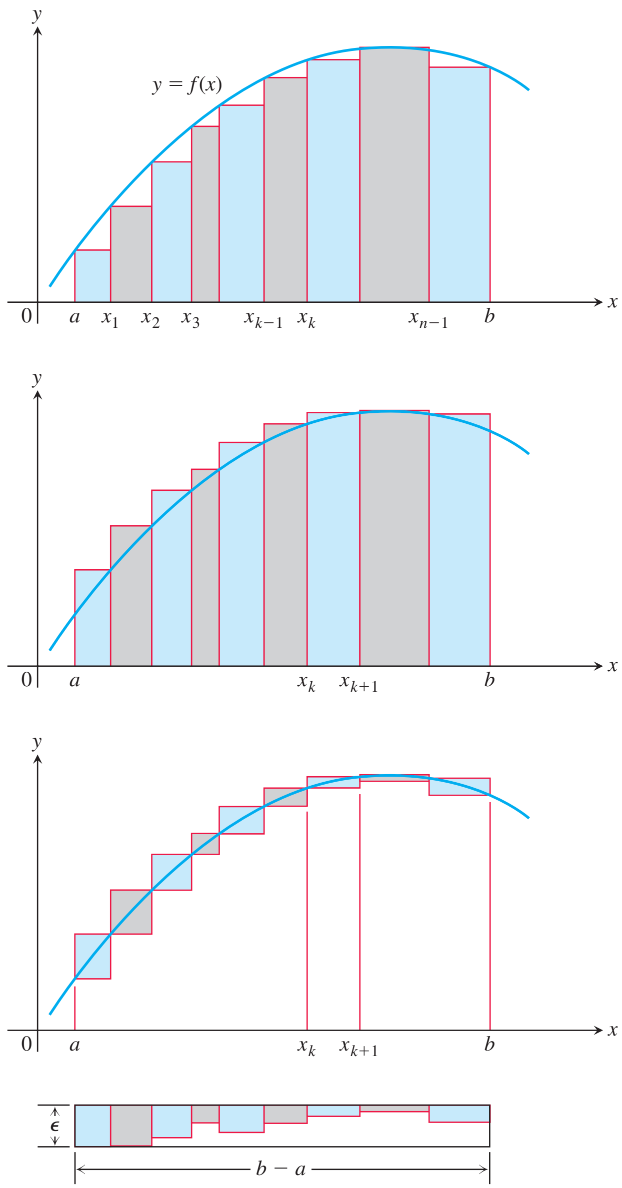
and the shaded regions in the first part of the figure.

b. If $M_k = \max\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **upper sum**

$$U = M_1\,\Delta x_1 + M_2\,\Delta x_2 + \cdots + M_n\,\Delta x_n$$

and the shaded regions in the second part of the figure.

c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.



87. We say f is **uniformly continuous** on $[a, b]$ if given any $\epsilon > 0$, there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if f is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \le \epsilon \cdot (b - a)$ by making the largest of the Δx_k's sufficiently small.

88. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer.

COMPUTER EXPLORATIONS

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 89–94. Use $n = 4$, 10, 20, and 50 subintervals of equal length in each case.

89. $\displaystyle\int_0^1 (1 - x)\,dx = \frac{1}{2}$

90. $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

91. $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = 0$

92. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = 1$

93. $\displaystyle\int_{-1}^{1} |x|\,dx = 1$

94. $\displaystyle\int_1^2 \frac{1}{x}dx$ (The integral's value is about 0.693.)

In Exercises 95–102, use a CAS to perform the following steps:

a. Plot the functions over the given interval.

b. Partition the interval into $n = 100$, 200, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) =$ (average value) for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

95. $f(x) = \sin x$ on $[0, \pi]$

96. $f(x) = \sin^2 x$ on $[0, \pi]$

97. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

98. $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

99. $f(x) = xe^{-x}$ on $[0, 1]$

100. $f(x) = e^{-x^2}$ on $[0, 1]$

101. $f(x) = \frac{\ln x}{x}$ on $[2, 5]$

102. $f(x) = \frac{1}{\sqrt{1 - x^2}}$ on $\left[0, \frac{1}{2}\right]$

5.4 The Fundamental Theorem of Calculus

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann sums as we did in Section 5.3. Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we present an integral version of the Mean Value Theorem, which is another important theorem of integral calculus and is used to prove the Fundamental Theorem. We also find that the net change of a function over an interval is the integral of its rate of change, as suggested by Example 3 in Section 5.1.

HISTORICAL BIOGRAPHY
Sir Isaac Newton
(1642–1727)

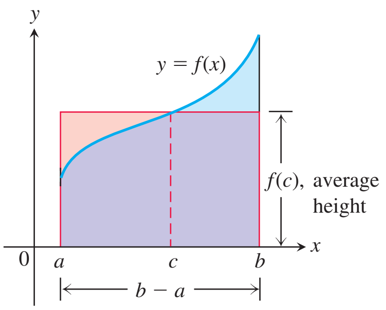


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\,dx.$$

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over a closed interval $[a, b]$ as the definite integral $\int_a^b f(x)\,dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.6, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b - a}\int_a^b f(x)\,dx \leq \max f.$$