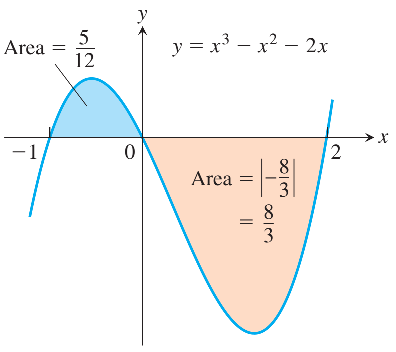


FIGURE 5.22 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

EXAMPLE 8 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.22). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$ ■

Exercises 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–34.

1. $\int_0^2 x(x-3)\,dx$
2. $\int_{-1}^{1} (x^2 - 2x + 3)\,dx$
3. $\int_{-2}^{2} \frac{3}{(x+3)^4}\,dx$
4. $\int_{-1}^{1} x^{299}\,dx$
5. $\int_1^4 \left(3x^2 - \frac{x^3}{4}\right) dx$
6. $\int_{-2}^{3} (x^3 - 2x + 3)\,dx$
7. $\int_0^1 (x^2 + \sqrt{x})\,dx$
8. $\int_1^{32} x^{-6/5}\,dx$
9. $\int_0^{\pi/3} 2\sec^2 x\,dx$
10. $\int_0^{\pi} (1 + \cos x)\,dx$
11. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$
12. $\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u}\,du$
13. $\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2}\,dt$
14. $\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt$
15. $\int_0^{\pi/4} \tan^2 x\,dx$
16. $\int_0^{\pi/6} (\sec x + \tan x)^2\,dx$
17. $\int_0^{\pi/8} \sin 2x\,dx$
18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$
19. $\int_1^{-1} (r+1)^2\,dr$
20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\,dt$
21. $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$
22. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\,dy$
23. $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\,ds$
24. $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}}\,dx$
25. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$
26. $\int_0^{\pi/3} (\cos x + \sec x)^2\,dx$
27. $\int_{-4}^{4} |x|\,dx$
28. $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$
29. $\int_0^{\ln 2} e^{3x}\,dx$
30. $\int_1^2 \left(\frac{1}{x} - e^{-x}\right) dx$
31. $\int_0^{1/2} \frac{4}{\sqrt{1 - x^2}}\,dx$
32. $\int_0^{1/\sqrt{3}} \frac{dx}{1 + 4x^2}$
33. $\int_2^4 x^{\pi - 1}\,dx$
34. $\int_{-1}^{0} \pi^{x-1}\,dx$

In Exercises 35–38, guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral. (*Hint:* Keep in mind the Chain Rule in guessing an antiderivative. You will learn how to find such antiderivatives in the next section.)

35. $\int_0^1 xe^{x^2}\,dx$
36. $\int_1^2 \frac{\ln x}{x}\,dx$
37. $\int_2^5 \frac{x\,dx}{\sqrt{1 + x^2}}$
38. $\int_0^{\pi/3} \sin^2 x\cos x\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 39–44.

a. by evaluating the integral and differentiating the result.

b. by differentiating the integral directly.

39. $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \cos t\, dt$

40. $\dfrac{d}{dx}\displaystyle\int_1^{\sin x} 3t^2\, dt$

41. $\dfrac{d}{dt}\displaystyle\int_0^{t^4} \sqrt{u}\, du$

42. $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan\theta} \sec^2 y\, dy$

43. $\dfrac{d}{dx}\displaystyle\int_0^{x^3} e^{-t}\, dt$

44. $\dfrac{d}{dt}\displaystyle\int_0^{\sqrt{t}} \left(x^4 + \dfrac{3}{\sqrt{1-x^2}}\right) dx$

Find dy/dx in Exercises 45–56.

45. $y = \displaystyle\int_0^x \sqrt{1+t^2}\, dt$

46. $y = \displaystyle\int_1^x \frac{1}{t}\, dt, \quad x > 0$

47. $y = \displaystyle\int_{\sqrt{x}}^0 \sin\left(t^2\right) dt$

48. $y = x\displaystyle\int_2^{x^2} \sin\left(t^3\right) dt$

49. $y = \displaystyle\int_{-1}^x \frac{t^2}{t^2+4}\, dt - \int_3^x \frac{t^2}{t^2+4}\, dt$

50. $y = \left(\displaystyle\int_0^x \left(t^3+1\right)^{10} dt\right)^3$

51. $y = \displaystyle\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}$

52. $y = \displaystyle\int_{\tan x}^0 \frac{dt}{1+t^2}$

53. $y = \displaystyle\int_0^{e^{x^2}} \frac{1}{\sqrt{t}}\, dt$

54. $y = \displaystyle\int_{2^x}^1 \sqrt[3]{t}\, dt$

55. $y = \displaystyle\int_0^{\sin^{-1} x} \cos t\, dt$

56. $y = \displaystyle\int_{-1}^{x^{1/\pi}} \sin^{-1} t\, dt$

Area

In Exercises 57–60, find the total area between the region and the x-axis.

57. $y = -x^2 - 2x, \quad -3 \le x \le 2$

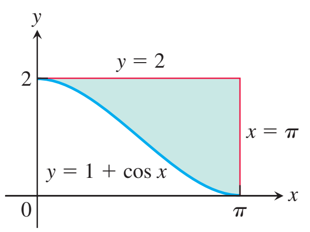
58. $y = 3x^2 - 3, \quad -2 \le x \le 2$

59. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

60. $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 61–64.

61.



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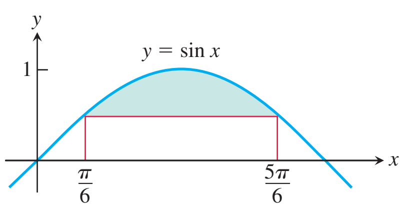

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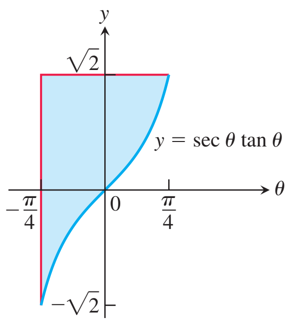

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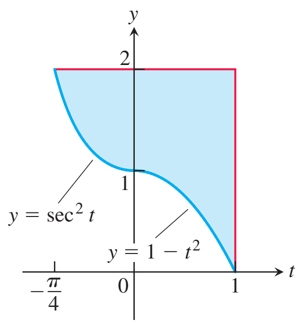


Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 65–68. Which function solves which problem? Give brief reasons for your answers.

a. $y = \displaystyle\int_1^x \frac{1}{t}\, dt - 3$

b. $y = \displaystyle\int_0^x \sec t\, dt + 4$

c. $y = \displaystyle\int_{-1}^x \sec t\, dt + 4$

d. $y = \displaystyle\int_\pi^x \frac{1}{t}\, dt - 3$

65. $\dfrac{dy}{dx} = \dfrac{1}{x}, \quad y(\pi) = -3$

66. $y' = \sec x, \quad y(-1) = 4$

67. $y' = \sec x, \quad y(0) = 4$

68. $y' = \dfrac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 69 and 70 in terms of integrals.

69. $\dfrac{dy}{dx} = \sec x, \quad y(2) = 3$

70. $\dfrac{dy}{dx} = \sqrt{1+x^2}, \quad y(1) = -2$

Theory and Examples

71. **Archimedes' area formula for parabolic arches** Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

72. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

73. **Cost from marginal cost** The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

74. **Revenue from marginal revenue** Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - 2/(x+1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

75. The temperature $T(°\text{F})$ of a room at time t minutes is given by

$$T = 85 - 3\sqrt{25 - t} \quad \text{for} \quad 0 \le t \le 25.$$

a. Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$.

b. Find the room's average temperature for $0 \le t \le 25$.

76. The height H (ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t + 1} + 5t^{1/3} \quad \text{for} \quad 0 \le t \le 8.$$

a. Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b. Find the tree's average height for $0 \le t \le 8$.

77. Suppose that $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

78. Find $f(4)$ if $\int_0^x f(t)\,dt = x\cos \pi x$.

79. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt$$

at $x = 1$.

80. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t - 1)\,dt$$

at $x = -1$.

81. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of dg/dx crosses the x-axis at $x = 1$.

82. Another proof of the Evaluation Theorem

a. Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$, and let F be any antiderivative of f. Show that

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

b. Apply the Mean Value Theorem to each term to show that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ for some c_i in the interval (x_{i-1}, x_i). Then show that $F(b) - F(a)$ is a Riemann sum for f on $[a, b]$.

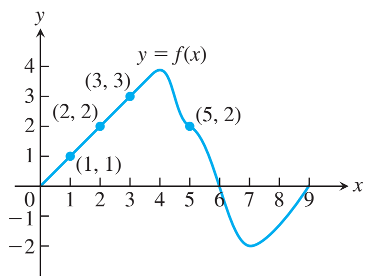
c. From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

83. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at time $t = 5$?

b. Is the acceleration of the particle at time $t = 5$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. At what time during the first 9 sec does s have its largest value?

e. Approximately when is the acceleration zero?

f. When is the particle moving toward the origin? Away from the origin?

g. On which side of the origin does the particle lie at time $t = 9$?

84. Find $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}} \int_1^x \frac{dt}{\sqrt{t}}$.

COMPUTER EXPLORATIONS

In Exercises 85–88, let $F(x) = \int_a^x f(t)\,dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

a. Plot the functions f and F together over $[a, b]$.

b. Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.

c. Over what intervals (approximately) is the function F increasing and decreasing? What is true about f over those intervals?

d. Calculate the derivative f' and plot it together with F. What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

85. $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

86. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \quad \left[0, \frac{9}{2}\right]$

87. $f(x) = \sin 2x \cos \frac{x}{3}, \quad [0, 2\pi]$

88. $f(x) = x\cos \pi x, \quad [0, 2\pi]$