Method 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2+1,\ du = 2z\,dz.$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2+1.$$

Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2+1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$

Exercises 5.5

Evaluating Indefinite Integrals

Evaluate the indefinite integrals in Exercises 1–16 by using the given substitutions to reduce the integrals to standard form.

1. $\int 2(2x+4)^5\,dx,\quad u = 2x+4$
2. $\int 7\sqrt{7x-1}\,dx,\quad u = 7x-1$
3. $\int 2x(x^2+5)^{-4}\,dx,\quad u = x^2+5$
4. $\int \frac{4x^3}{(x^4+1)^2}\,dx,\quad u = x^4+1$
5. $\int (3x+2)(3x^2+4x)^4\,dx,\quad u = 3x^2+4x$
6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx,\quad u = 1+\sqrt{x}$
7. $\int \sin 3x\,dx,\quad u = 3x$
8. $\int x\sin(2x^2)\,dx,\quad u = 2x^2$
9. $\int \sec 2t\tan 2t\,dt,\quad u = 2t$
10. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt,\quad u = 1-\cos\frac{t}{2}$
11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}},\quad u = 1-r^3$
12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy,\quad u = y^4+4y^2+1$
13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx,\quad u = x^{3/2}-1$
14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx,\quad u = -\frac{1}{x}$
15. $\int \csc^2 2\theta\cot 2\theta\,d\theta$
    a. Using $u = \cot 2\theta$
    b. Using $u = \csc 2\theta$
16. $\int \frac{dx}{\sqrt{5x+8}}$
    a. Using $u = 5x+8$
    b. Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 17–66.

17. $\int \sqrt{3-2s}\,ds$
18. $\int \frac{1}{\sqrt{5s+4}}\,ds$
19. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$
20. $\int 3y\sqrt{7-3y^2}\,dy$
21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$
22. $\int \sqrt{\sin x}\cos^3 x\,dx$

23. $\int \sec^2(3x+2)\,dx$

24. $\int \tan^2 x \sec^2 x\,dx$

25. $\int \sin^5 \frac{x}{3}\cos\frac{x}{3}\,dx$

26. $\int \tan^7 \frac{x}{2}\sec^2\frac{x}{2}\,dx$

27. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

28. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

29. $\int x^{1/2}\sin\left(x^{3/2}+1\right)dx$

30. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

31. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

32. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

33. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

34. $\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt$

35. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

36. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

37. $\int \frac{x}{\sqrt{1+x}}\,dx$

38. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

39. $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

41. $\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx$

42. $\int \sqrt{\frac{x^4}{x^3-1}}\,dx$

43. $\int x(x-1)^{10}\,dx$

44. $\int x\sqrt{4-x}\,dx$

45. $\int (x+1)^2(1-x)^5\,dx$

46. $\int (x+5)(x-5)^{1/3}\,dx$

47. $\int x^3\sqrt{x^2+1}\,dx$

48. $\int 3x^5\sqrt{x^3+1}\,dx$

49. $\int \frac{x}{(x^2-4)^3}\,dx$

50. $\int \frac{x}{(2x-1)^{2/3}}\,dx$

51. $\int (\cos x)\,e^{\sin x}\,dx$

52. $\int (\sin 2\theta)\,e^{\sin^2\theta}\,d\theta$

53. $\int \frac{1}{\sqrt{x}e^{-\sqrt{x}}}\sec^2\left(e^{\sqrt{x}}+1\right)dx$

54. $\int \frac{1}{x^2}e^{1/x}\sec\left(1+e^{1/x}\right)\tan\left(1+e^{1/x}\right)dx$

55. $\int \frac{dx}{x\ln x}$

56. $\int \frac{\ln\sqrt{t}}{t}\,dt$

57. $\int \frac{dz}{1+e^z}$

58. $\int \frac{dx}{x\sqrt{x^4-1}}$

59. $\int \frac{5}{9+4r^2}\,dr$

60. $\int \frac{1}{\sqrt{e^{2\theta}-1}}\,d\theta$

61. $\int \frac{e^{\sin^{-1}x}\,dx}{\sqrt{1-x^2}}$

62. $\int \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1-x^2}}$

63. $\int \frac{(\sin^{-1}x)^2\,dx}{\sqrt{1-x^2}}$

64. $\int \frac{\sqrt{\tan^{-1}x}\,dx}{1+x^2}$

65. $\int \frac{dy}{(\tan^{-1}y)(1+y^2)}$

66. $\int \frac{dy}{(\sin^{-1}y)\sqrt{1-y^2}}$

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 67 and 68.

67. $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

a. $u=\tan x$, followed by $v=u^3$, then by $w=2+v$

b. $u=\tan^3 x$, followed by $v=2+u$

c. $u=2+\tan^3 x$

68. $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

a. $u=x-1$, followed by $v=\sin u$, then by $w=1+v^2$

b. $u=\sin(x-1)$, followed by $v=1+u^2$

c. $u=1+\sin^2(x-1)$

Evaluate the integrals in Exercises 69 and 70.

69. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

70. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

71. Find the integral of $\cot x$ using a substitution like that in Example 7c.

72. Find the integral of $\csc x$ by multiplying by an appropriate form equal to 1, as in Example 8b.

Initial Value Problems

Solve the initial value problems in Exercises 73–78.

73. $\frac{ds}{dt}=12t\,(3t^2-1)^3,\quad s(1)=3$

74. $\frac{dy}{dx}=4x\,(x^2+8)^{-1/3},\quad y(0)=0$

75. $\frac{ds}{dt}=8\sin^2\left(t+\frac{\pi}{12}\right),\quad s(0)=8$

76. $\frac{dr}{d\theta}=3\cos^2\left(\frac{\pi}{4}-\theta\right),\quad r(0)=\frac{\pi}{8}$

77. $\frac{d^2s}{dt^2}=-4\sin\left(2t-\frac{\pi}{2}\right),\quad s'(0)=100,\quad s(0)=0$

78. $\frac{d^2y}{dx^2}=4\sec^2 2x\tan 2x,\quad y'(0)=4,\quad y(0)=-1$

79. The velocity of a particle moving back and forth on a line is $v=ds/dt=6\sin 2t$ m/sec for all t. If $s=0$ when $t=0$, find the value of s when $t=\pi/2$ sec.

80. The acceleration of a particle moving back and forth on a line is $a=d^2s/dt^2=\pi^2\cos\pi t$ m/sec² for all t. If $s=0$ and $v=8$ m/sec when $t=0$, find s when $t=1$ sec.