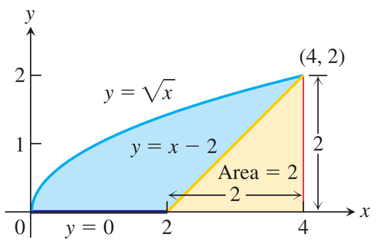


FIGURE 5.31 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Although it was easier to find the area in Example 6 by integrating with respect to y rather than x (just as we did in Example 7), there is an easier way yet. Looking at Figure 5.31, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x-axis for $0 \le x \le 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\begin{aligned}\text{Area} &= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2)\\ &= \frac{2}{3}x^{3/2}\Big]_0^4 - 2\\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.\end{aligned}$$

Exercises 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–46.

1. a. $\displaystyle\int_0^3 \sqrt{y+1}\,dy$ **b.** $\displaystyle\int_{-1}^0 \sqrt{y+1}\,dy$

2. a. $\displaystyle\int_0^1 r\sqrt{1-r^2}\,dr$ **b.** $\displaystyle\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a. $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x\,dx$ **b.** $\displaystyle\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a. $\displaystyle\int_0^{\pi} 3\cos^2 x \sin x\,dx$ **b.** $\displaystyle\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a. $\displaystyle\int_0^1 t^3(1+t^4)^3\,dt$ **b.** $\displaystyle\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a. $\displaystyle\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ **b.** $\displaystyle\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a. $\displaystyle\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ **b.** $\displaystyle\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a. $\displaystyle\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ **b.** $\displaystyle\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a. $\displaystyle\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ **b.** $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ **b.** $\displaystyle\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a. $\displaystyle\int_0^1 t\sqrt{4+5t}\,dt$ **b.** $\displaystyle\int_1^9 t\sqrt{4+5t}\,dt$

12. a. $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$ **b.** $\displaystyle\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

13. a. $\displaystyle\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ **b.** $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a. $\displaystyle\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$ **b.** $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

15. $\displaystyle\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$

16. $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

17. $\displaystyle\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

18. $\displaystyle\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\displaystyle\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$

20. $\displaystyle\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\displaystyle\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

22. $\displaystyle\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

23. $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$

24. $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

25. $\displaystyle\int_0^{\pi/4} (1+e^{\tan\theta})\sec^2\theta\,d\theta$

26. $\displaystyle\int_{\pi/4}^{\pi/2} (1+e^{\cot\theta})\csc^2\theta\,d\theta$

27. $\displaystyle\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$

28. $\displaystyle\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

29. $\displaystyle\int_1^2 \frac{2\ln x}{x}\,dx$

30. $\displaystyle\int_2^4 \frac{dx}{x\ln x}$

31. $\displaystyle\int_2^4 \frac{dx}{x(\ln x)^2}$

32. $\displaystyle\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

33. $\displaystyle\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

34. $\displaystyle\int_{\pi/4}^{\pi/2} \cot t\,dt$

35. $\displaystyle\int_0^{\pi/3} \tan^2\theta\cos\theta\,d\theta$

36. $\displaystyle\int_0^{\pi/12} 6\tan 3x\,dx$

37. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1 + (\sin\theta)^2}$

38. $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1 + (\cot x)^2}$

39. $\displaystyle\int_{0}^{\ln\sqrt{3}} \frac{e^x\, dx}{1 + e^{2x}}$

40. $\displaystyle\int_{1}^{e^{\pi/4}} \frac{4\, dt}{t(1 + \ln^2 t)}$

41. $\displaystyle\int_{0}^{1} \frac{4\, ds}{\sqrt{4 - s^2}}$

42. $\displaystyle\int_{0}^{\sqrt[3]{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

43. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$

44. $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$

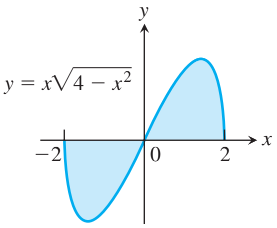
45. $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$

46. $\displaystyle\int_{0}^{3} \frac{y\, dy}{\sqrt{5y + 1}}$

Area

Find the total areas of the shaded regions in Exercises 47–62.

47.



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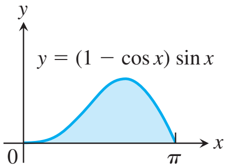

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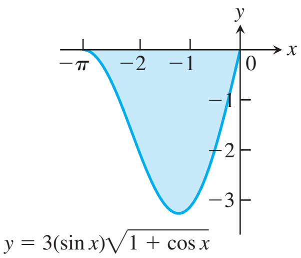
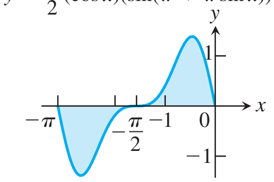


50. $y = \dfrac{\pi}{2}(\cos x)(\sin(\pi + \pi\sin x))$



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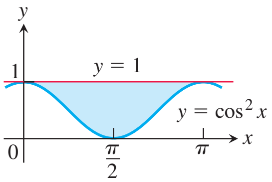

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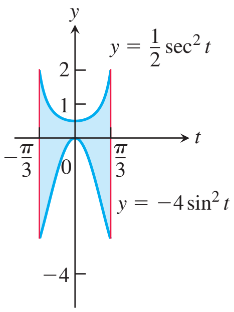

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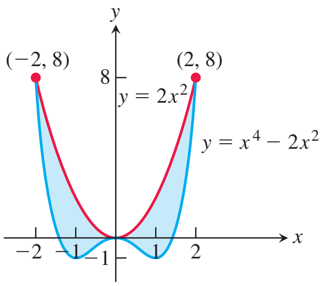

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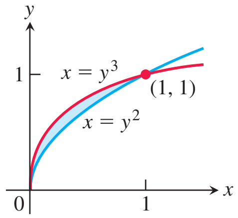
54.



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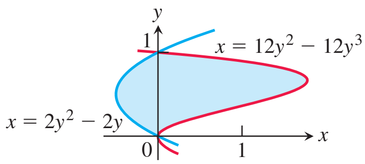

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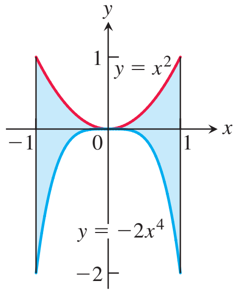

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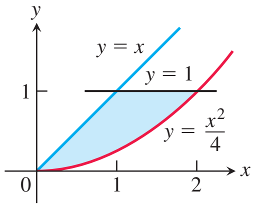

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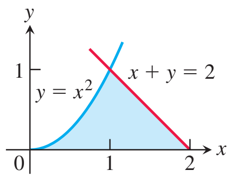

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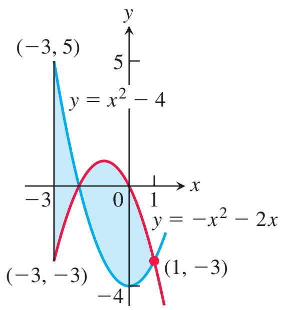

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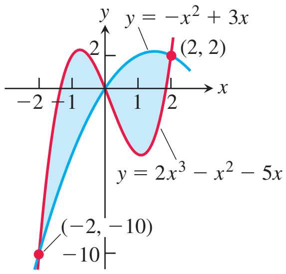

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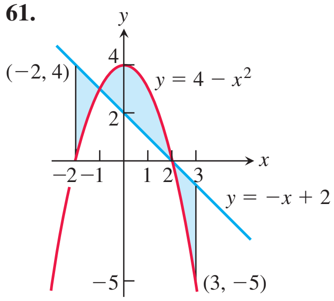

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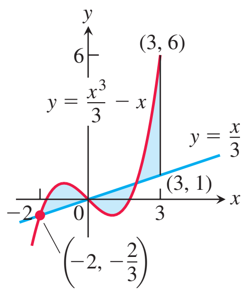


Find the areas of the regions enclosed by the lines and curves in Exercises 63–72.

63. $y = x^2 - 2$ and $y = 2$

64. $y = 2x - x^2$ and $y = -3$

65. $y = x^4$ and $y = 8x$

66. $y = x^2 - 2x$ and $y = x$

67. $y = x^2$ and $y = -x^2 + 4x$

68. $y = 7 - 2x^2$ and $y = x^2 + 4$

69. $y = x^4 - 4x^2 + 4$ and $y = x^2$

70. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

71. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

72. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 73–80.

73. $x = 2y^2$, $x = 0$, and $y = 3$

74. $x = y^2$ and $x = y + 2$

75. $y^2 - 4x = 4$ and $4x - y = 16$

76. $x - y^2 = 0$ and $x + 2y^2 = 3$

77. $x + y^2 = 0$ and $x + 3y^2 = 2$

78. $x - y^{2/3} = 0$ and $x + y^4 = 2$

79. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

80. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 81–84.

81. $4x^2 + y = 4$ and $x^4 - y = 1$

82. $x^3 - y = 0$ and $3x^2 - y = 4$

83. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

84. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 85–92.

85. $y = 2\sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

86. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

87. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

88. $y = \sin(\pi x/2)$ and $y = x$

89. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

90. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

91. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

92. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

Area Between Curves

93. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

94. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

95. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

96. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

97. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

98. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

99. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

100. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

101. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \leq x \leq 2$ of the x-axis.

102. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \leq x \leq 1$ of the x-axis.

103. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

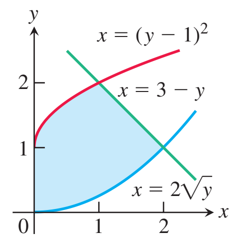
b. Find c by integrating with respect to y. (This puts c in the limits of integration.)

c. Find c by integrating with respect to x. (This puts c into the integrand as well.)

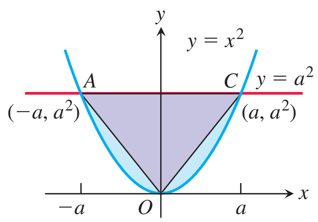
104. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

105. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

106. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



107. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

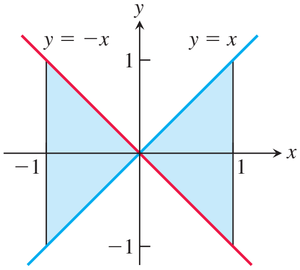


108. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

109. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\int_{-1}^{1} (x - (-x))\, dx = \int_{-1}^{1} 2x\, dx$

b. $\int_{-1}^{1} (-x - (x))\, dx = \int_{-1}^{1} -2x\, dx$



110. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_{a}^{b} [\, f(x) - g(x)\,]\, dx.$$

Give reasons for your answer.

Theory and Examples

111. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_{1}^{3} \frac{\sin 2x}{x}\, dx$$

in terms of F.

112. Show that if f is continuous, then

$$\int_{0}^{1} f(x)\, dx = \int_{0}^{1} f(1 - x)\, dx.$$

113. Suppose that

$$\int_{0}^{1} f(x)\, dx = 3.$$

Find

$$\int_{-1}^{0} f(x)\, dx$$

if **a.** f is odd, **b.** f is even.

114. a. Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\, dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

115. If f is a continuous function, find the value of the integral

$$I = \int_{0}^{a} \frac{f(x)\, dx}{f(x) + f(a - x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

116. By using a substitution, prove that for all positive numbers x and y,

$$\int_{x}^{xy} \frac{1}{t}\, dt = \int_{1}^{y} \frac{1}{t}\, dt.$$

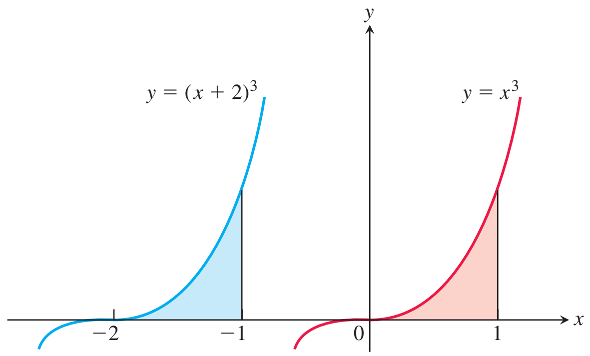
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_{a}^{b} f(x)\, dx = \int_{a-c}^{b-c} f(x + c)\, dx. \qquad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x + 2)^3\, dx = \int_{0}^{1} x^3\, dx$$

because the areas of the shaded regions are congruent.



117. Use a substitution to verify Equation (1).

118. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x + c)$ over $[a - c, b - c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x) = x^2, \quad a = 0, \quad b = 1, \quad c = 1$

b. $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad c = \pi/2$

c. $f(x) = \sqrt{x - 4}, \quad a = 4, \quad b = 8, \quad c = 5$

COMPUTER EXPLORATIONS

In Exercises 119–122, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a. Plot the curves together to see what they look like and how many points of intersection they have.

b. Use the numerical equation solver in your CAS to find all the points of intersection.

c. Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

d. Sum together the integrals found in part (c).

119. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}, \quad g(x) = x - 1$

120. $f(x) = \frac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

121. $f(x) = x + \sin(2x), \quad g(x) = x^3$

122. $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$