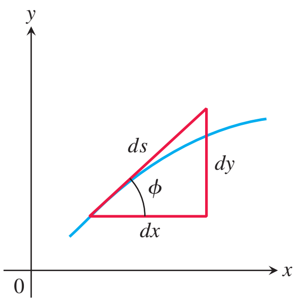


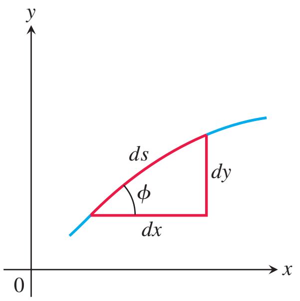




**FIGURE 6.27** Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

$L = \int ds$. Figure 6.27a gives the exact interpretation of $ds$ corresponding to Equation (7). Figure 6.27b is not strictly accurate, but is to be thought of as a simplified approximation of Figure 6.27a. That is, $ds \approx \Delta s$.

**EXAMPLE 5** Find the arc length function for the curve in Example 2, taking $A = (1, 13/12)$ as the starting point (see Figure 6.25).

**Solution** In the solution to Example 2, we found that

$$1 + [f'(x)]^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

Therefore the arc length function is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt$$

$$= \left[\frac{t^3}{12} - \frac{1}{t}\right]_1^x = \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}.$$

To compute the arc length along the curve from $A = (1, 13/12)$ to $B = (4, 67/12)$, for instance, we simply calculate

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12} = 6.$$

This is the same result we obtained in Example 2. ■

## Exercises 6.3

### Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–14. If you have a grapher, you may want to graph these curves to see what they look like.

**1.** $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

**2.** $y = x^{3/2}$ from $x = 0$ to $x = 4$

**3.** $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$

**4.** $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$

**5.** $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$

**6.** $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$

**7.** $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5,\quad 1 \le x \le 8$

**8.** $y = (x^3/3) + x^2 + x + 1/(4x + 4),\quad 0 \le x \le 2$

**9.** $y = \ln x - \dfrac{x^2}{8}$ from $x = 1$ to $x = 2$

**10.** $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$ from $x = 1$ to $x = 3$

**11.** $y = \dfrac{x^3}{3} + \dfrac{1}{4x},\quad 1 \le x \le 3$

**12.** $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3},\quad \dfrac{1}{2} \le x \le 1$

**13.** $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}\, dt,\quad -\pi/4 \le y \le \pi/4$

**14.** $y = \displaystyle\int_{-2}^x \sqrt{3t^4 - 1}\, dt,\quad -2 \le x \le -1$

### T Finding Integrals for Lengths of Curves

In Exercises 15–22, do the following.

**a.** Set up an integral for the length of the curve.

**b.** Graph the curve to see what it looks like.

**c.** Use your grapher's or computer's integral evaluator to find the curve's length numerically.

**15.** $y = x^2,\quad -1 \le x \le 2$

**16.** $y = \tan x,\quad -\pi/3 \le x \le 0$

**17.** $x = \sin y,\quad 0 \le y \le \pi$

**18.** $x = \sqrt{1 - y^2},\quad -1/2 \le y \le 1/2$

**19.** $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$

**20.** $y = \sin x - x\cos x,\quad 0 \le x \le \pi$

**21.** $y = \displaystyle\int_0^x \tan t\, dt,\quad 0 \le x \le \pi/6$

**22.** $x = \displaystyle\int_0^y \sqrt{\sec^2 t - 1}\, dt,\quad -\pi/3 \le y \le \pi/4$

### Theory and Examples

**23. a.** Find a curve with a positive derivative through the point (1, 1) whose length integral (Equation 3) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

**b.** How many such curves are there? Give reasons for your answer.

**24.** **a.** Find a curve with a positive derivative through the point (0, 1) whose length integral (Equation 4) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

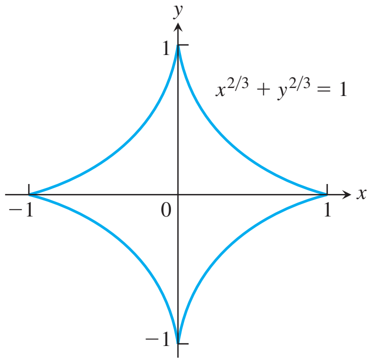
**b.** How many such curves are there? Give reasons for your answer.

**25.** Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$.

**26. The length of an astroid** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.




**27. Length of a line segment** Use the arc length formula (Equation 3) to find the length of the line segment $y = 3 - 2x$, $0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

**28. Circumference of a circle** Set up an integral to find the circumference of a circle of radius $r$ centered at the origin. You will learn how to evaluate the integral in Section 8.4.

**29.** If $9x^2 = y(y - 3)^2$, show that

$$ds^2 = \frac{(y + 1)^2}{4y}\, dy^2.$$

**30.** If $4x^2 - y^2 = 64$, show that

$$ds^2 = \frac{4}{y^2}\left(5x^2 - 16\right) dx^2.$$

**31.** Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

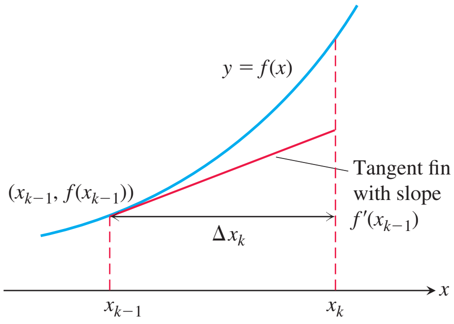
**32. Using tangent fins to derive the length formula for curves** Assume that $f$ is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.

**a.** Show that the length of the $k$th tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\,\Delta x_k)^2}$.

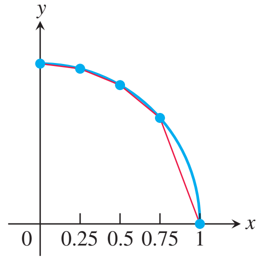
**b.** Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

which is the length $L$ of the curve $y = f(x)$ from $a$ to $b$.




**33.** Approximate the arc length of one-quarter of the unit circle (which is $\pi/2$) by computing the length of the polygonal approximation with $n = 4$ segments (see accompanying figure).




**34. Distance between two points** Assume that the two points $(x_1, y_1)$ and $(x_2, y_2)$ lie on the graph of the straight line $y = mx + b$. Use the arc length formula (Equation 3) to find the distance between the two points.

**35.** Find the arc length function for the graph of $f(x) = 2x^{3/2}$ using (0, 0) as the starting point. What is the length of the curve from (0, 0) to (1, 2)?

**36.** Find the arc length function for the curve in Exercise 8, using (0, 1/4) as the starting point. What is the length of the curve from (0, 1/4) to (1, 59/24)?

## COMPUTER EXPLORATIONS

In Exercises 37–42, use a CAS to perform the following steps for the given graph of the function over the closed interval.

**a.** Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.22.)

**b.** Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

**c.** Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as $n$ increases? Explain your answer.

**37.** $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

**38.** $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

**39.** $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

**40.** $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

**41.** $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$

**42.** $f(x) = x^3 - x^2, \quad -1 \le x \le 1$