THEOREM 2—Pappus's Theorem for Surface Areas If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{11}$$

The proof we give assumes that we can model the axis of revolution as the x-axis and the arc as the graph of a continuously differentiable function of x.

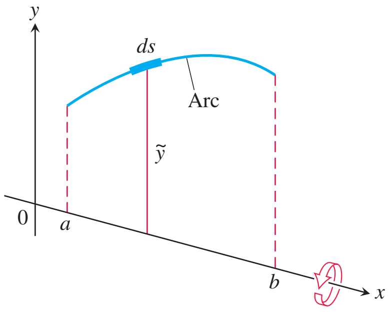


FIGURE 6.59 Figure for proving Pappus's Theorem for surface area. The arc length differential ds is given by Equation (6) in Section 6.3.

Proof We draw the axis of revolution as the x-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Figure 6.59). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y \, ds = 2\pi \int_{x=a}^{x=b} y \, ds. \tag{12}$$

The y-coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y} \, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y \, ds}{L}. \qquad L = \int ds \text{ is the arc's length and } \tilde{y} = y.$$

Hence

$$\int_{x=a}^{x=b} y \, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Equation (12) gives $S = 2\pi\bar{y}L$. With ρ equal to $\bar{y}$, we have $S = 2\pi\rho L$. ■

EXAMPLE 8 Use Pappus's area theorem to find the surface area of the torus in Example 6.

Solution From Figure 6.57, the surface of the torus is generated by revolving a circle of radius a about the z-axis, and $b \geq a$ is the distance from the centroid to the axis of revolution. The arc length of the smooth curve generating this surface of revolution is the circumference of the circle, so $L = 2\pi a$. Substituting these values into Equation (11), we find the surface area of the torus to be

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba. \qquad ■$$

Exercises 6.6

Thin Plates with Constant Density

In Exercises 1–14, find the center of mass of a thin plate of constant density δ covering the given region.

1. The region bounded by the parabola $y = x^2$ and the line $y = 4$
2. The region bounded by the parabola $y = 25 - x^2$ and the x-axis
3. The region bounded by the parabola $y = x - x^2$ and the line $y = -x$
4. The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$
5. The region bounded by the y-axis and the curve $x = y - y^3$, $0 \leq y \leq 1$

6. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$

7. The region bounded by the x-axis and the curve $y = \cos x$, $-\pi/2 \leq x \leq \pi/2$

8. The region between the curve $y = \sec^2 x$, $-\pi/4 \leq x \leq \pi/4$ and the x-axis

T **9.** The region between the curve $y = 1/x$ and the x-axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.

10. a. The region cut from the first quadrant by the circle $x^2 + y^2 = 9$

b. The region bounded by the x-axis and the semicircle $y = \sqrt{9 - x^2}$

Compare your answer in part (b) with the answer in part (a).

11. The region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$

12. The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$

13. The region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$

14. The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a\to\infty} \bar{x}$.

Thin Plates with Varying Density

15. Find the center of mass of a thin plate covering the region between the x-axis and the curve $y = 2/x^2$, $1 \leq x \leq 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.

16. Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.

17. The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y-axis to generate a solid.

a. Find the volume of the solid.

b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.

c. Sketch the plate and show the center of mass in your sketch.

18. The region between the curve $y = 2/x$ and the x-axis from $x = 1$ to $x = 4$ is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.

c. Sketch the plate and show the center of mass in your sketch.

Centroids of Triangles

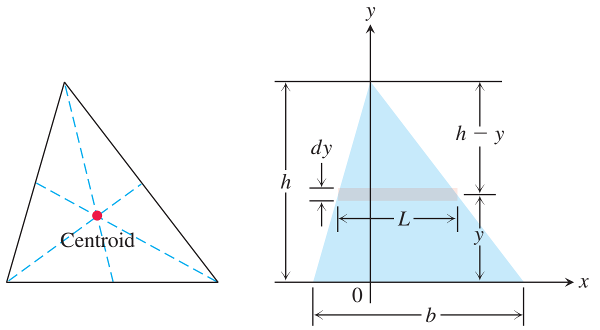
19. The centroid of a triangle lies at the intersection of the triangle's medians You may recall that the point inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.

i) Stand one side of the triangle on the x-axis as in part (b) of the accompanying figure. Express dm in terms of L and dy.

ii) Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for L in your formula for dm.

iii) Show that $\bar{y} = h/3$.

iv) Extend the argument to the other sides.



(a) (b)

Use the result in Exercise 19 to find the centroids of the triangles whose vertices appear in Exercises 20–24. Assume $a, b > 0$.

20. $(-1, 0)$, $(1, 0)$, $(0, 3)$

21. $(0, 0)$, $(1, 0)$, $(0, 1)$

22. $(0, 0)$, $(a, 0)$, $(0, a)$

23. $(0, 0)$, $(a, 0)$, $(0, b)$

24. $(0, 0)$, $(a, 0)$, $(a/2, b)$

Thin Wires

25. Constant density Find the moment about the x-axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.

26. Constant density Find the moment about the x-axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.

27. Variable density Suppose that the density of the wire in Example 4 is $\delta = k \sin\theta$ (k constant). Find the center of mass.

28. Variable density Suppose that the density of the wire in Example 4 is $\delta = 1 + k|\cos\theta|$ (k constant). Find the center of mass.

Plates Bounded by Two Curves

In Exercises 29–32, find the centroid of the thin plate bounded by the graphs of the given functions. Use Equations (6) and (7) with $\delta = 1$ and $M =$ area of the region covered by the plate.

29. $g(x) = x^2$ and $f(x) = x + 6$

30. $g(x) = x^2(x + 1)$, $f(x) = 2$, and $x = 0$

31. $g(x) = x^2(x - 1)$ and $f(x) = x^2$

32. $g(x) = 0$, $f(x) = 2 + \sin x$, $x = 0$, and $x = 2\pi$

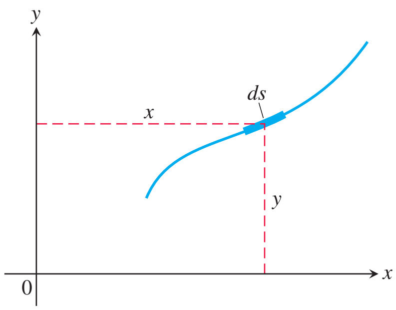
(*Hint:* $\int x \sin x\, dx = \sin x - x\cos x + C$.)

Theory and Examples

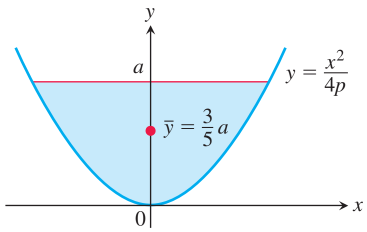
Verify the statements and formulas in Exercises 33 and 34.

33. The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x\, ds}{\text{length}}, \qquad \bar{y} = \frac{\int y\, ds}{\text{length}}.$$



34. Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the y-coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.



The Theorems of Pappus

35. The square region with vertices (0, 2), (2, 0), (4, 2), and (2, 4) is revolved about the x-axis to generate a solid. Find the volume and surface area of the solid.

36. Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$ (see Exercise 19).

37. Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the y-axis.

38. Use the theorems of Pappus to find the lateral surface area and the volume of a right-circular cone.

39. Use Pappus's Theorem for surface area and the fact that the surface area of a sphere of radius a is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

40. As found in Exercise 39, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

41. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x-axis is $(1/2)\pi ab$, and the volume of the ellipsoid generated by revolving R about the x-axis is $(4/3)\pi ab^2$. Find the centroid of R. Notice that the location is independent of a.

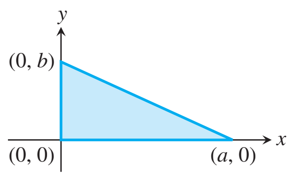
42. As found in Example 7, the centroid of the region enclosed by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

43. The region of Exercise 42 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

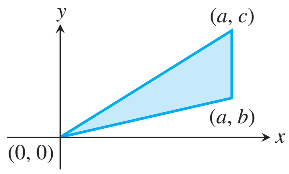
44. As found in Exercise 39, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

In Exercises 45 and 46, use a theorem of Pappus to find the centroid of the given triangle. Use the fact that the volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

45.



46.



Chapter 6 Questions to Guide Your Review

1. How do you define and calculate the volumes of solids by the method of slicing? Give an example.
2. How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
3. Describe the method of cylindrical shells. Give an example.
4. How do you find the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
5. How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x)$, $a \le x \le b$, about the x-axis? Give an example.
6. How do you define and calculate the work done by a variable force directed along a portion of the x-axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.
7. How do you calculate the force exerted by a liquid against a portion of a flat vertical wall? Give an example.
8. What is a center of mass? a centroid?
9. How do you locate the center of mass of a thin flat plate of material? Give an example.
10. How do you locate the center of mass of a thin plate bounded by two curves $y = f(x)$ and $y = g(x)$ over $a \le x \le b$?