$$\frac{1}{\pi}\int_0^\pi x^3 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{n}\sin nx + \frac{3x^3}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx\right]_0^\pi$$

$$= \frac{1}{\pi}\left(\frac{3\pi^2\cos n\pi}{n^2} - \frac{6\cos n\pi}{n^4} + \frac{6}{n^4}\right)$$

$$= \frac{3}{\pi}\left(\frac{\pi^2 n^2(-1)^n + 2(-1)^{n+1} + 2}{n^4}\right). \qquad \cos n\pi = (-1)^n$$

Integrals like those in Example 8 occur frequently in electrical engineering.

Exercises 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\,dx$
2. $\int \theta \cos \pi\theta\,d\theta$
3. $\int t^2 \cos t\,dt$
4. $\int x^2 \sin x\,dx$
5. $\int_1^2 x \ln x\,dx$
6. $\int_1^e x^3 \ln x\,dx$
7. $\int xe^x\,dx$
8. $\int xe^{3x}\,dx$
9. $\int x^2e^{-x}\,dx$
10. $\int (x^2 - 2x + 1)e^{2x}\,dx$
11. $\int \tan^{-1} y\,dy$
12. $\int \sin^{-1} y\,dy$
13. $\int x \sec^2 x\,dx$
14. $\int 4x \sec^2 2x\,dx$
15. $\int x^3e^x\,dx$
16. $\int p^4e^{-p}\,dp$
17. $\int (x^2 - 5x)e^x\,dx$
18. $\int (r^2 + r + 1)e^r\,dr$
19. $\int x^5e^x\,dx$
20. $\int t^2e^{4t}\,dt$
21. $\int e^\theta \sin\theta\,d\theta$
22. $\int e^{-y}\cos y\,dy$
23. $\int e^{2x}\cos 3x\,dx$
24. $\int e^{-2x}\sin 2x\,dx$

Using Substitution

Evaluate the integrals in Exercise 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$
26. $\int_0^1 x\sqrt{1-x}\,dx$
27. $\int_0^{\pi/3} x\tan^2 x\,dx$
28. $\int \ln(x + x^2)\,dx$
29. $\int \sin(\ln x)\,dx$
30. $\int z(\ln z)^2\,dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–52. Some integrals do not require integration by parts.

31. $\int x\sec x^2\,dx$
32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$
33. $\int x(\ln x)^2\,dx$
34. $\int \frac{1}{x(\ln x)^2}\,dx$
35. $\int \frac{\ln x}{x^2}\,dx$
36. $\int \frac{(\ln x)^3}{x}\,dx$
37. $\int x^3e^{x^4}\,dx$
38. $\int x^5e^{x^3}\,dx$
39. $\int x^3\sqrt{x^2+1}\,dx$
40. $\int x^2\sin x^3\,dx$
41. $\int \sin 3x\cos 2x\,dx$
42. $\int \sin 2x\cos 4x\,dx$
43. $\int \sqrt{x}\ln x\,dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
45. $\int \cos\sqrt{x}\,dx$
46. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$
47. $\int_0^{\pi/2} \theta^2\sin 2\theta\,d\theta$
48. $\int_0^{\pi/2} x^3\cos 2x\,dx$
49. $\int_{2/\sqrt{3}}^2 t\sec^{-1} t\,dt$
50. $\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\,dx$
51. $\int x\tan^{-1} x\,dx$
52. $\int x^2\tan^{-1}\frac{x}{2}\,dx$

Theory and Examples

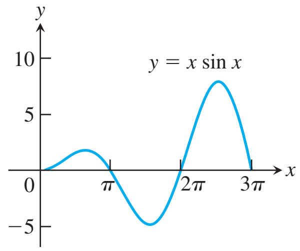
53. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$.

b. $\pi \le x \le 2\pi$.

c. $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



54. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for

a. $\pi/2 \le x \le 3\pi/2$.

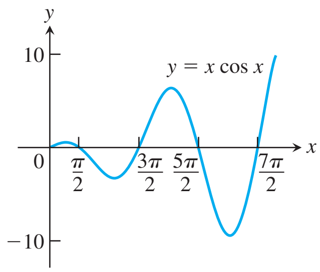
b. $3\pi/2 \le x \le 5\pi/2$.

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



55. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

56. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis.

b. about the line $x = 1$.

57. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis.

b. the line $x = \pi/2$.

58. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about

a. the y-axis.

b. the line $x = \pi$.

(See Exercise 53 for a graph.)

59. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the x-axis.

c. Find the volume of the solid formed by revolving this region about the line $x = -2$.

d. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.

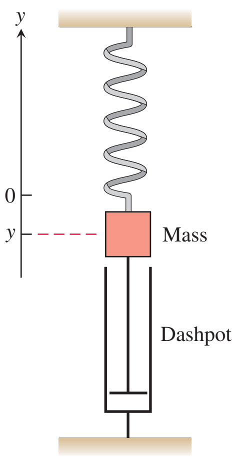
a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the y-axis.

61. Average value A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



62. Average value In a mass-spring-dashpot system like the one in Exercise 61, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 63–67, use integration by parts to establish the reduction formula.

63. $\displaystyle\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

64. $\displaystyle\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$

65. $\displaystyle\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

66. $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

67. $\displaystyle\int x^m(\ln x)^n\,dx = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\cdot$

$\displaystyle\int x^m(\ln x)^{n-1}\,dx, \quad m \neq -1$

68. Use Example 5 to show that

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases} \left(\dfrac{\pi}{2}\right)\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n \text{ even} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n \text{ odd} \end{cases}$$

69. Show that

$$\int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)f(x)\,dx.$$

70. Use integration by parts to obtain the formula

$$\int\sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned}\int f^{-1}(x)\,dx &= \int y f'(y)\,dy && y = f^{-1}(x),\quad x = f(y),\ dx = f'(y)\,dy \\ &= yf(y) - \int f(y)\,dy && \text{Integration by parts with } u = y,\ dv = f'(y)\,dy \\ &= xf^{-1}(x) - \int f(y)\,dy\end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned}\int \ln x\,dx &= \int ye^y\,dy && y = \ln x,\quad x = e^y,\ dx = e^y\,dy \\ &= ye^y - e^y + C \\ &= x\ln x - x + C.\end{aligned}$$

For the integral of $\cos^{-1}x$ we get

$$\begin{aligned}\int \cos^{-1}x\,dx &= x\cos^{-1}x - \int \cos y\,dy && y = \cos^{-1}x \\ &= x\cos^{-1}x - \sin y + C \\ &= x\cos^{-1}x - \sin(\cos^{-1}x) + C.\end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 71–74. Express your answers in terms of x.

71. $\displaystyle\int \sin^{-1}x\,dx$

72. $\displaystyle\int \tan^{-1}x\,dx$

73. $\displaystyle\int \sec^{-1}x\,dx$

74. $\displaystyle\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \tag{5}$$

Exercises 75 and 76 compare the results of using Equations (4) and (5).

75. Equations (4) and (5) give different formulas for the integral of $\cos^{-1}x$:

a. $\displaystyle\int \cos^{-1}x\,dx = x\cos^{-1}x - \sin(\cos^{-1}x) + C$ Eq. (4)

b. $\displaystyle\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

76. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1}x$:

a. $\displaystyle\int \tan^{-1}x\,dx = x\tan^{-1}x - \ln\sec(\tan^{-1}x) + C$ Eq. (4)

b. $\displaystyle\int \tan^{-1}x\,dx = x\tan^{-1}x - \ln\sqrt{1+x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 77 and 78 with (**a**) Eq. (4) and (**b**) Eq. (5). In each case, check your work by differentiating your answer with respect to x.

77. $\displaystyle\int \sinh^{-1}x\,dx$

78. $\displaystyle\int \tanh^{-1}x\,dx$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\,dx = \tan x + C.$$