Exercises 8.4

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int \frac{dx}{\sqrt{9 + x^2}}$

2. $\displaystyle\int \frac{3\,dx}{\sqrt{1 + 9x^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4 + x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8 + 2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9 - x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1 - 4x^2}}$

7. $\displaystyle\int \sqrt{25 - t^2}\,dt$

8. $\displaystyle\int \sqrt{1 - 9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2 - 49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2 - 9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2 - 49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2 - 25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - 1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2 - 1}}, \quad x > 1$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\displaystyle\int \frac{x}{\sqrt{9 - x^2}}\,dx$

16. $\displaystyle\int \frac{x^2}{4 + x^2}\,dx$

17. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2 + 4}}$

18. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 + 1}}$

19. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4 - w^2}}$

20. $\displaystyle\int \frac{\sqrt{9 - w^2}}{w^2}\,dw$

21. $\displaystyle\int \sqrt{\frac{x + 1}{1 - x}}\,dx$

22. $\displaystyle\int x\sqrt{x^2 - 4}\,dx$

23. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1 - x^2)^{3/2}}$

24. $\displaystyle\int_{0}^{1} \frac{dx}{(4 - x^2)^{3/2}}$

25. $\displaystyle\int \frac{dx}{(x^2 - 1)^{3/2}}, \quad x > 1$

26. $\displaystyle\int \frac{x^2\,dx}{(x^2 - 1)^{5/2}}, \quad x > 1$

27. $\displaystyle\int \frac{(1 - x^2)^{3/2}}{x^6}\,dx$

28. $\displaystyle\int \frac{(1 - x^2)^{1/2}}{x^4}\,dx$

29. $\displaystyle\int \frac{8\,dx}{(4x^2 + 1)^2}$

30. $\displaystyle\int \frac{6\,dt}{(9t^2 + 1)^2}$

31. $\displaystyle\int \frac{x^3\,dx}{x^2 - 1}$

32. $\displaystyle\int \frac{x\,dx}{25 + 4x^2}$

33. $\displaystyle\int \frac{v^2\,dv}{(1 - v^2)^{5/2}}$

34. $\displaystyle\int \frac{(1 - r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

35. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t} + 9}}$

36. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1 + e^{2t})^{3/2}}$

37. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t} + 4t\sqrt{t}}$

38. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1 + (\ln y)^2}}$

39. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$

40. $\displaystyle\int \frac{dx}{1 + x^2}$

41. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 - 1}}$

42. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}}$

43. $\displaystyle\int \frac{x\,dx}{\sqrt{1 + x^4}}$

44. $\displaystyle\int \frac{\sqrt{1 - (\ln x)^2}}{x \ln x}\,dx$

45. $\displaystyle\int \sqrt{\frac{4 - x}{x}}\,dx$

(*Hint:* Let $x = u^2$.)

46. $\displaystyle\int \sqrt{\frac{x}{1 - x^3}}\,dx$

(*Hint:* Let $u = x^{3/2}$.)

47. $\displaystyle\int \sqrt{x}\,\sqrt{1 - x}\,dx$

48. $\displaystyle\int \frac{\sqrt{x - 2}}{\sqrt{x - 1}}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 49–52 for y as a function of x.

49. $x\dfrac{dy}{dx} = \sqrt{x^2 - 4}, \quad x \ge 2, \quad y(2) = 0$

50. $\sqrt{x^2 - 9}\,\dfrac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

51. $(x^2 + 4)\dfrac{dy}{dx} = 3, \quad y(2) = 0$

52. $(x^2 + 1)^2\dfrac{dy}{dx} = \sqrt{x^2 + 1}, \quad y(0) = 1$

Applications and Examples

53. Area Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2}/3$.

54. Area Find the area enclosed by the ellipse

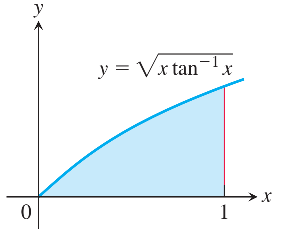
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

55. Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.

a. Find the area of the region.

b. Find the centroid of the region.

56. Consider the region bounded by the graphs of $y = \sqrt{x \tan^{-1} x}$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the x-axis (see accompanying figure).



57. Evaluate $\int x^3 \sqrt{1 - x^2}\, dx$ using

a. integration by parts.

b. a u-substitution.

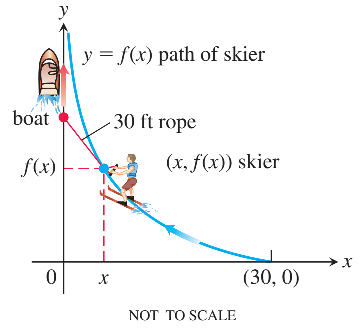
c. a trigonometric substitution.

58. Path of a water skier Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 ft long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.

a. Show that $f'(x) = \dfrac{-\sqrt{900 - x^2}}{x}$.

(*Hint:* Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)

b. Solve the equation in part (a) for $f(x)$, using $f(30) = 0$.



8.5 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}.$$

You can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x^2 - 2x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\begin{aligned}\int \frac{5x - 3}{(x + 1)(x - 3)}\, dx &= \int \frac{2}{x + 1}\, dx + \int \frac{3}{x - 3}\, dx \\ &= 2 \ln|x + 1| + 3 \ln|x - 3| + C.\end{aligned}$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}. \tag{1}$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x + 1)$ and $B/(x - 3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until suitable values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.