**EXAMPLE 9** Find $A$, $B$, and $C$ in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to $x$.

**Solution** Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find $A$, $B$, and $C$:

$$\begin{aligned}
x = 1: \quad & (1)^2 + 1 = A(-1)(-2) + B(0) + C(0)\\
& 2 = 2A\\
& A = 1\\
x = 2: \quad & (2)^2 + 1 = A(0) + B(1)(-1) + C(0)\\
& 5 = -B\\
& B = -5\\
x = 3: \quad & (3)^2 + 1 = A(0) + B(0) + C(2)(1)\\
& 10 = 2C\\
& C = 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

## Exercises 8.5

### Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$

2. $\dfrac{5x-7}{x^2-3x+2}$

3. $\dfrac{x+4}{(x+1)^2}$

4. $\dfrac{2x+2}{x^2-2x+1}$

5. $\dfrac{z+1}{z^2(z-1)}$

6. $\dfrac{z}{z^3-z^2-6z}$

7. $\dfrac{t^2+8}{t^2-5t+6}$

8. $\dfrac{t^4+9}{t^4+9t^2}$

### Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$

10. $\displaystyle\int \frac{dx}{x^2+2x}$

11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$

12. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$

14. $\displaystyle\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3+t^2-2t}$

16. $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

### Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

19. $\displaystyle\int \frac{dx}{(x^2-1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

### Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$

23. $\displaystyle\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$

25. $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$

26. $\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds$

27. $\displaystyle\int \frac{x^2-x+2}{x^3-1}\,dx$

28. $\displaystyle\int \frac{1}{x^4+x}\,dx$

29. $\displaystyle\int \frac{x^2}{x^4-1}\,dx$

30. $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}\,dx$

31. $\displaystyle\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$

32. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

### Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

**33.** $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

**34.** $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

**35.** $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

**36.** $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

**37.** $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

**38.** $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

### Evaluating Integrals

Evaluate the integrals in Exercises 39–50.

**39.** $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

**40.** $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

**41.** $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

**42.** $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

**43.** $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x-2)^2}\,dx$

**44.** $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2 + 1)(x+1)^2}\,dx$

**45.** $\displaystyle\int \frac{1}{x^{3/2} - \sqrt{x}}\,dx$

**46.** $\displaystyle\int \frac{1}{(x^{1/3} - 1)\sqrt{x}}\,dx$
($Hint:$ Let $x = u^6$.)

**47.** $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$
($Hint:$ Let $x + 1 = u^2$.)

**48.** $\displaystyle\int \frac{1}{x\sqrt{x+9}}\,dx$

**49.** $\displaystyle\int \frac{1}{x(x^4 + 1)}\,dx$
$\left(\textit{Hint:}\text{ Multiply by } \frac{x^3}{x^3}.\right)$

**50.** $\displaystyle\int \frac{1}{x^6(x^5 + 4)}\,dx$

### Initial Value Problems

Solve the initial value problems in Exercises 51–54 for $x$ as a function of $t$.

**51.** $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

**52.** $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

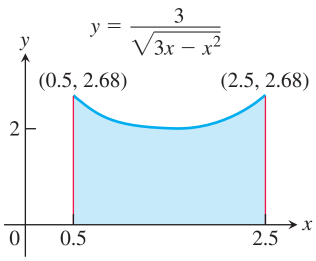
**53.** $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

**54.** $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

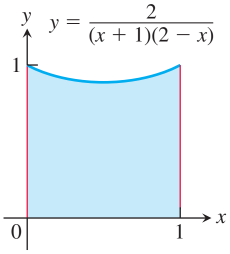
### Applications and Examples

In Exercises 55 and 56, find the volume of the solid generated by revolving the shaded region about the indicated axis.

**55.** The $x$-axis

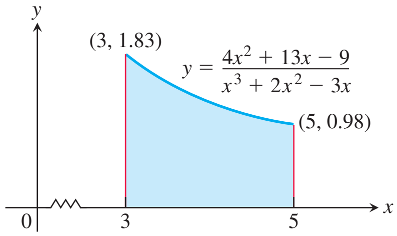



**56.** The $y$-axis




**57.** Find, to two decimal places, the $x$-coordinate of the centroid of the region in the first quadrant bounded by the $x$-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

**58.** Find the $x$-coordinate of the centroid of this region to two decimal places.




**59. Social diffusion** Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people $x$ who have the information is treated as a differentiable function of time $t$, and the rate of diffusion, $dx/dt$, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where $N$ is the number of people in the population.

Suppose $t$ is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

**a.** Find $x$ as a function of $t$.

**b.** When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

**60. Second-order chemical reactions** Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If $a$ is the amount of substance $A$ and $b$ is the amount of substance $B$ at time $t = 0$, and if $x$ is the amount of product at time $t$, then the rate of formation of $x$ may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k,$$

where $k$ is a constant for the reaction. Integrate both sides of this equation to obtain a relation between $x$ and $t$ **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.