Exercises 8.8

Evaluating Improper Integrals

The integrals in Exercises 1–34 converge. Evaluate the integrals without using tables.

1. $\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\int_1^{\infty} \frac{dx}{x^{1.001}}$

3. $\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\int_0^1 \frac{dr}{r^{0.999}}$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

11. $\int_2^{\infty} \frac{2}{v^2-v}\,dv$

12. $\int_2^{\infty} \frac{2\,dt}{t^2-1}$

13. $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

14. $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

18. $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\int_0^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\int_{-\infty}^0 \theta e^{\theta}\,d\theta$

22. $\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta$

23. $\int_{-\infty}^0 e^{-|x|}\,dx$

24. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

25. $\int_0^1 x\ln x\,dx$

26. $\int_0^1 (-\ln x)\,dx$

27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6}$

34. $\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_0^{\pi/2} \tan\theta\,d\theta$

36. $\int_0^{\pi/2} \cot\theta\,d\theta$

37. $\int_0^1 \frac{\ln x}{x^2}\,dx$

38. $\int_1^2 \frac{dx}{x\ln x}$

39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

41. $\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$

42. $\int_0^1 \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

43. $\int_0^2 \frac{dx}{1-x^2}$

44. $\int_0^2 \frac{dx}{1-x}$

45. $\int_{-1}^1 \ln|x|\,dx$

46. $\int_{-1}^1 -x\ln|x|\,dx$

47. $\int_1^{\infty} \frac{dx}{x^3+1}$

48. $\int_4^{\infty} \frac{dx}{\sqrt{x}-1}$

49. $\int_2^{\infty} \frac{dv}{\sqrt{v-1}}$

50. $\int_0^{\infty} \frac{d\theta}{1+e^{\theta}}$

51. $\int_0^{\infty} \frac{dx}{\sqrt{x^6+1}}$

52. $\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$

53. $\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$

54. $\int_2^{\infty} \frac{x\,dx}{\sqrt{x^4-1}}$

55. $\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$

56. $\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$

57. $\int_4^{\infty} \frac{2\,dt}{t^{3/2}-1}$

58. $\int_2^{\infty} \frac{1}{\ln x}\,dx$

59. $\int_1^{\infty} \frac{e^x}{x}\,dx$

60. $\int_{e^e}^{\infty} \ln(\ln x)\,dx$

61. $\int_1^{\infty} \frac{1}{\sqrt{e^x-x}}\,dx$

62. $\int_1^{\infty} \frac{1}{e^x-2^x}\,dx$

63. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$

64. $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

Theory and Examples

65. Find the values of p for which each integral converges.

a. $\int_1^2 \frac{dx}{x(\ln x)^p}$ **b.** $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$

66. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$ Show that

$$\int_0^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

67. Find the area of the region.

68. Find the centroid of the region.

69. Find the volume of the solid generated by revolving the region about the y-axis.

70. Find the volume of the solid generated by revolving the region about the x-axis.

71. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

72. The region in Exercise 71 is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

73. Evaluate the integrals.

a. $\displaystyle\int_0^1 \frac{dt}{\sqrt{t}(1 + t)}$ **b.** $\displaystyle\int_0^\infty \frac{dt}{\sqrt{t}(1 + t)}$

74. Evaluate $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x^2 - 9}}$.

75. Estimating the value of a convergent improper integral whose domain is infinite

a. Show that

$$\int_3^\infty e^{-3x}\,dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

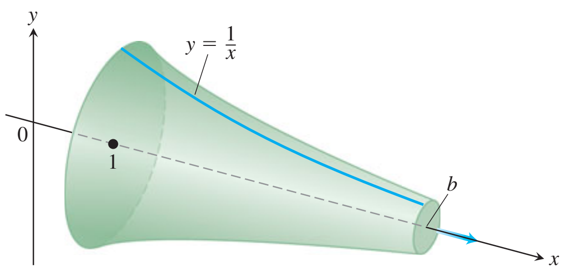
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

76. The infinite paint can or Gabriel's horn As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx > 2\pi \int_1^b \frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges.

a. Calculate it.

b. This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

77. Sine-integral function The integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

called the *sine-integral function*, has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t > 0$. Is the sine-integral function everywhere increasing or decreasing? Do you think $\mathrm{Si}(x) = 0$ for $x > 0$? Check your answers by graphing the function $\mathrm{Si}(x)$ for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

78. Error function The function

$$\mathrm{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.4, Exercise 41.

79. Normal probability distribution The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean. (See Section 8.9.)

From the theory of probability, it is known that

$$\int_{-\infty}^\infty f(x)\,dx = 1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

T **a.** Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

b. Evaluate

$$\int_{-n}^n f(x)\,dx$$

for $n = 1, 2$, and 3.

c. Give a convincing argument that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_{b}^{\infty} e^{-x/2}\,dx \to 0 \quad \text{as} \quad b \to \infty.)$$

80. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{b}^{\infty} f(x)\,dx$ both converge.

b. $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$ when the integrals involved converge.

COMPUTER EXPLORATIONS

In Exercises 81–84, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

81. $\displaystyle\int_{0}^{e} x^{p} \ln x\,dx$

82. $\displaystyle\int_{e}^{\infty} x^{p} \ln x\,dx$

83. $\displaystyle\int_{0}^{\infty} x^{p} \ln x\,dx$

84. $\displaystyle\int_{-\infty}^{\infty} x^{p} \ln |x|\,dx$

Use a CAS to evaluate the integrals.

85. $\displaystyle\int_{0}^{2/\pi} \sin \frac{1}{x}\,dx$

86. $\displaystyle\int_{0}^{2/\pi} x \sin \frac{1}{x}\,dx$

8.9 Probability

The outcome of some events, such as a heavy rock falling from a great height, can be modeled so that we can predict with high accuracy what will happen. On the other hand, many events have more than one possible outcome and which one of them will occur is uncertain. If we toss a coin, a head or a tail will result with each outcome being equally likely, but we do not know in advance which one it will be. If we randomly select and then weigh a person from a large population, there are many possible weights the person might have, and it is not certain whether the weight will be between 180 and 190 lb. We are told it is highly likely, but not known for sure, that an earthquake of magnitude 6.0 or greater on the Richter scale will occur near a major population area in California within the next one hundred years. Events having more than one possible outcome are *probabilistic* in nature, and when modeling them we assign a *probability* to the likelihood that a particular outcome may occur. In this section we show how calculus plays a central role in making predictions with probabilistic models.

Random Variables

We begin our discussion with some familiar examples of uncertain events for which the collection of all possible outcomes is finite.

EXAMPLE 1

(a) If we toss a coin once, there are two possible outcomes $\{\text{H, T}\}$, where H represents the coin landing head face up and T a tail landing face up. If we toss a coin three times, there are eight possible outcomes, taking into account the order in which a head or tail occurs. The set of outcomes is $\{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}$.

(b) If we roll a six-sided die once, the set of possible outcomes is $\{1, 2, 3, 4, 5, 6\}$ representing the six faces of the die.

(c) If we select at random two cards from a 52-card deck, there are 52 possible outcomes for the first card drawn and then 51 possibilities for the second card. Since the order of the cards does not matter, there are $(52 \cdot 51)/2 = 1{,}326$ possible outcomes altogether. ■

It is customary to refer to the set of all possible outcomes as the *sample space* for an event. With an uncertain event we are usually interested in which outcomes, if any, are more likely to occur than others, and to how large an extent. In tossing a coin three times,