Chen-Stein method for Poisson: key estimate

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MAT 235A

November 25, 2025

Stein's equation

Let P_λ denote the Poisson probability function with parameter λ, that is $P_\lambda(k) = e^{-\lambda}\lambda^k/k!$ for $k \geq 0$, and $P_\lambda(A) = \sum_{k \in A} P_\lambda(k)$.

Fix a set $A \subset \mathbb{Z}_+$. Then the *Stein's equation* for the function $f_A : \mathbb{Z}_+ \to \mathbb{R}$ is

$$\begin{aligned} 1_{\{k \in A\}} - P_\lambda(A) &= \lambda f_A(k+1) - k f_A(k), \quad k = 0, 1, \ldots \\ f_A(0) &= 0. \end{aligned} \tag{1}$$

Note that f_A is uniquely determined by (1).

Stein's equation

Let L be the operator on functions $f : \mathbb{Z}_+ \to \mathbb{R}$, given by $Lf(k) = \lambda f(k+1) - kf(k)$, $k \geq 0$. This is a linear operator. Furthermore, let $g_A(k) = 1_{\{k \in A\}} - P_\lambda(A)$. Then f_A is the unique function f that solves $Lf = g_A$ and satisfies $f(0) = 0$. Note that L is linear, and

$$\begin{aligned} g_{A \cup B} &= g_A + g_B \quad \text{if } A \cap B = \emptyset, \\ g_{A^c} &= -g_A. \end{aligned}$$

Then, by linearity of L,

$$\begin{aligned} f_{A \cup B} &= f_A + f_B \quad \text{if } A \cap B = \emptyset, \\ f_{A^c} &= -f_A. \end{aligned} \tag{2}$$

Solution to Stein's equation

From the equation

$$f_A(k+1) = \frac{1}{\lambda} 1_{\{k \in A\}} - \frac{1}{\lambda} P_\lambda(A) + \frac{1}{\lambda} k f_A(k)$$

we get

$$f_A(1) = \frac{1}{\lambda} 1_{\{0 \in A\}} - \frac{1}{\lambda} P_\lambda(A)$$
$$f_A(2) = \frac{1}{\lambda} 1_{\{1 \in A\}} + \frac{1}{\lambda^2} 1_{\{0 \in A\}} - \left(\frac{1}{\lambda} + \frac{1}{\lambda^2} \right) P_\lambda(A)$$
$$f_A(3) = \frac{1}{\lambda} 1_{\{2 \in A\}} + \frac{2}{\lambda^2} 1_{\{1 \in A\}} + \frac{2}{\lambda^3} 1_{\{0 \in A\}} - \left(\frac{1}{\lambda} + \frac{2}{\lambda^2} + \frac{2}{\lambda^3} \right) P_\lambda(A)$$

Solution to Stein's equation

By induction, for any $k \geq 0$,

$$
\begin{aligned}
& f_A(k+1) \\
&= \frac{1}{\lambda} 1_{\{k \in A\}} + \frac{k}{\lambda^2} 1_{\{k-1 \in A\}} + \frac{k(k-1)}{\lambda^3} 1_{\{k-2 \in A\}} + \cdots + \frac{k!}{\lambda^{k+1}} 1_{\{0 \in A\}} \\
&\quad - \left(\frac{1}{\lambda} + \frac{k}{\lambda^2} + \frac{k(k-1)}{\lambda^3} + \ldots + \frac{k!}{\lambda^{k+1}} \right) P_\lambda(A) \\
&= \frac{k!}{\lambda^{k+1}} \left(\sum_{i=0}^{k} \frac{\lambda^i}{i!} 1_{\{i \in A\}} - P_\lambda(A) \sum_{i=0}^{k} \frac{\lambda^i}{i!} \right).
\end{aligned}
$$

Solution to Stein's equation

Let $U_k = \{0, 1, \ldots, k\}$.

$$\begin{aligned} & f_A(k+1) \\ &= \frac{k!}{\lambda^{k+1}} \left(\sum_{i=0}^{k} \frac{\lambda^i}{i!} 1_{\{i \in A\}} - P_\lambda(A) \sum_{i=0}^{k} \frac{\lambda^i}{i!} \right) \\ &= \frac{k!}{\lambda^{k+1}} e^\lambda [P_\lambda(A \cap U_k) - P_\lambda(A) P_\lambda(U_k)] \\ &= \frac{k!}{\lambda^{k+1}} e^\lambda [P_\lambda(A \cap U_k) - P_\lambda(A \cap U_k) P_\lambda(U_k) \\ &\qquad + P_\lambda(A \cap U_k) P_\lambda(U_k) - P_\lambda(A) P_\lambda(U_k)] \\ &= \frac{k!}{\lambda^{k+1}} e^\lambda [P_\lambda(A \cap U_k) P_\lambda(U_k^c) - P_\lambda(A \cap U_k^c) P_\lambda(U_k)]. \end{aligned} \tag{3}$$

Solution to Stein's equation

For $A \subset \mathbb{Z}_+$, write $A_n = A \cap U_n$ and $A'_n = A \setminus A_n$.

The second line of (3) says

$$f_A(k+1) = \frac{k!}{\lambda^{k+1}} e^{\lambda} [P_\lambda(A \cap U_k) - P_\lambda(A) P_\lambda(U_k)],$$

Apply this to $A = A'_n$: for large n, $A'_n \cap U_k = \emptyset$ and $P_\lambda(A'_n) \to 0$.

We get that, for every fixed k, $f_{A'_n}(k+1) \to 0$ as $n \to \infty$.

Therefore, by additivity (the first line of (2)),

$$f_{A_n}(k+1) \to f_A(k+1) \quad \text{as } n \to \infty, \tag{4}$$

pointwise in k.

Stein's equation: key bound

For $f : \mathbb{Z}_+ \to \mathbb{R}$, let

$$\Delta f = \sup\{|f(k+1) - f(k)| : k \geq 1\}.$$

Lemma

For any $A \subset \mathbb{Z}_+$,

$$\Delta f_A \leq \lambda^{-1}(1 - e^{-\lambda}) \leq \min(1, \lambda^{-1}).$$

Proof of the key bound

Proof.

First we claim that it suffices to prove that

$$f_A(k+1) - f_A(k) \leq \lambda^{-1}(1 - e^{-\lambda}), \tag{5}$$

for all A and $k \geq 1$. Indeed, if this holds, we can apply it to the complement, and the second line of (2) implies that

$$f_A(k+1) - f_A(k) = -(f_{A^c}(k+1) - f_{A^c}(k)) \geq -\lambda^{-1}(1 - e^{-\lambda}),$$

and thus $|f_A(k+1) - f_A(k)| \leq \lambda^{-1}(1 - e^{-\lambda})$.

Proof of the key bound

Proof, continued.

To prove (5) we may, by (4), assume that A is finite. In this case, using the abbreviation $f_j = f_{\{j\}}$, additivity (first line of (2)) implies

$$f_A = \sum_{j \in A} f_j. \tag{6}$$

Proof of the key bound

Proof, continued.

The second line of (3) says

$$f_A(k+1) = \frac{k!}{\lambda^{k+1}} e^{\lambda} [P_\lambda(A \cap U_k) - P_\lambda(A) P_\lambda(U_k)],$$

so

$$f_j(k+1) = \frac{k!}{\lambda^{k+1}} e^{\lambda} P_\lambda(j) [1_{\{j \leq k\}} - P_\lambda(U_k)]. \tag{7}$$

If $k \geq j$, then, by (7),

$$f_j(k+1) = \frac{1}{\lambda} P_\lambda(j) \sum_{i=1}^{\infty} \frac{\lambda^i}{(i+k)(i-1+k)\cdots(1+k)},$$

which is positive and decreasing in k.

Proof of the key bound

Proof, continued.

Now we use (7), which says

$$f_j(k+1) = \frac{k!}{\lambda^{k+1}} e^{\lambda} P_\lambda(j)[1_{\{j \le k\}} - P_\lambda(U_k)].$$

for $k < j$:

$$f_j(k+1) = -\frac{1}{\lambda} P_\lambda(j) \left(1 + \frac{k}{\lambda} + \frac{k(k-1)}{\lambda^2} + \cdots + \frac{k!}{\lambda^k}\right),$$

which is negative and decreasing in k. The only $k \ge 1$ for which $f_j(k+1) - f_j(k) \ge 0$ then is $k = j$.

Proof of the key bound

Proof, continued.

For $j \geq 1$, by the fourth line of (3),

$$\begin{aligned} f_j(j+1) - f_j(j) &= \frac{j!}{\lambda^{j+1}} e^{\lambda} P_\lambda(j) P_\lambda(U_j^c) + \frac{(j-1)!}{\lambda^j} e^{\lambda} P_\lambda(j) P_\lambda(U_{j-1}) \\ &= \frac{1}{\lambda} \sum_{i=j+1}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} + \frac{1}{j} \sum_{i=0}^{j-1} \frac{\lambda^i}{i!} e^{-\lambda} \\ &= \frac{e^{-\lambda}}{\lambda} \left(\sum_{i=j+1}^{\infty} \frac{\lambda^i}{i!} + \sum_{i=1}^{j} \frac{\lambda^i}{i!} \cdot \frac{i}{j} \right) \\ &\leq \frac{e^{-\lambda}}{\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} = \lambda^{-1}(1 - e^{-\lambda}). \end{aligned} \tag{8}$$

Proof of the key bound

Proof, continued.

For $j = 0$, we use that $f_0(k+1)$ is decreasing for all $k \geq 0$, so that $f_0(k+1) - f_0(k) \leq 0$ for $k \geq 1$.

Thus we have, by (6), for every A and $k \geq 1$,

$$\begin{aligned} f_A(k+1) - f_A(k) &= \sum_{j \in A} (f_j(k+1) - f_j(k)) \\ &\leq f_k(k+1) - f_k(k) \leq \lambda^{-1}(1 - e^{-\lambda}), \end{aligned}$$

which proves (5) and ends the proof. □

Setup for applications of Stein's equation

It is easy to show that, if W is a Poisson random variable with $EW = \lambda$, and f is any function on $\mathbb{Z}_+$ that grows at most exponentially (that is, $|f(x)| \leq \exp(Cx)$ for some constant C), $E[\lambda f(W+1) - Wf(W)] = 0$. It is reasonable to expect that, if a random variable W with values in $\mathbb{Z}_+$ is such that this equality approximately holds for a class of functions f, then W is approximately Poisson.

That class are functions f_A!

Note that, by the Lemma, f_A do not grow faster than linearly.

Setup for applications of Stein's equation

Assume W is a random variable with values in $\mathbb{Z}_+$. The essence of Chen-Stein method is that an estimate

$$E[\lambda f_A(W+1) - W f_A(W)] \le \alpha, \tag{9}$$

where α does not depend on A, immediately implies (as we can apply it to $f_{A^c} = -f_A$) the same bound for the absolute value and, as we will now check, for the total variation distance from P_λ.

Plug in $k = W$ into (1) to get

$$1_{\{W \in A\}} - P_\lambda(A) = \lambda f_A(W+1) - W f_A(W)$$

and then, by taking expectation,

$$\begin{aligned} d_{\mathrm{TV}}(W, P_\lambda) &= \sup_A |P(W \in A\} - P_\lambda(A)| \\ &= \sup_A |E[\lambda f_A(W+1) - W f_A(W)]| \le \alpha. \end{aligned}$$

Setup for applications of Stein's equation

To get (9) using the Lemma, one needs to produce Δf_A as a factor in an upper bound for $E[\lambda f_A(W+1) - W f_A(W)]$, multiplied by a small quantity.

This can be done in many cases when W is a sum of mildly dependent indicators. In fact, the estimate typically works with arbitrary f (satisfying the restriction on its growth), only using that $f = f_A$ at the end, when we need an estimate for Δf_A.