Math/Stat 235A, Fall 2025

Final Exam

Due: Monday, Dec. 8, 11:59pm, in Gradescope.

Directions: Work on these problems *alone*; you cannot discuss any part of the exam with anybody or use any books, papers, web sites, etc.; you can consult only your notes from 235A. Try to give concise solutions; none of the problems requires a long argument or computation. To facilitate grading, please solve each problem on a separate page, and select the corresponding problem number for each question when you upload to Gradescope. I will not reply to specific questions about exam problems. However, if you think that a problem is misstated, please let me know. You will receive extra credit if you are the first person to spot a mistake and I will post any corrections or clarifications on the course's web page.

1. (a) Assume that random variables X_n have $E|X_n| < \infty$ for all n and $\lim_n E|X_n| = 0$. Show that X_n are uniformly integrable. (*Hint*. Recall that sup in the definition of u.i. can be replaced by lim sup.)
(b) On $(\Omega, \mathcal{F}) = ([0,1], \mathcal{B}([0,1]))$, with P the Lebesgue measure, define the random variables

$$X_n = \frac{n}{\log n} 1_{(0,1/n)}$$

Show that X_n are uniformly integrable. Show that there is no random variable $Y \geq 0$ with $EY < \infty$ and $X_n \leq Y$ for all n. (*Hint*. Find the lower bound on Y on $(1/2^{k+1}, 1/2^k)$.)

2. Let U_n be i.i.d. random variables, uniform on $[0,1]$. For each sequence X_n of random variables below, show that it converges a.s., and that there exists a constant a and a deterministic sequence b_n so that $b_n(X_n - a)$ converges in distribution to a nontrivial (that is, not a.s. constant) random variable. Let $S_n = U_1 + \cdots + U_n$.
(a) $X_n = S_n/n$.
(b) $X_n = (U_1^2 + \cdots + U_n^2)/n$.
(c) $X_n = S_n/(U_1^2 + \cdots + U_n^2)$.
(d) $X_n = (U_1 \cdots U_n)^{1/n}$. (*Hint*: log.)
(e) $X_n = \sqrt{S_n(n - S_n)}/n$.

3. (a) Assume that X_n, $n = 1, 2, 3, \dots$ are independent random variables with values in $[0, b]$, where $b \in (0, \infty)$ is a nonrandom constant. Prove that $\sum_{n=1}^\infty X_n$ converges a.s. if and only if $\sum_{n=1}^\infty EX_n < \infty$.
(b) Assume that X with values in $[0,1]$ has the bounded Pareto distribution with parameter $\alpha > 0$, given by $P(X \geq x) = (1-x)^\alpha$ for $x \in [0,1]$. Show that $n^\alpha E(X^n)$ converges to a limit in $(0, \infty)$. (*Hint*. Use substitution $t = (1-x)n$.)
(c) Let now X_n be i.i.d. with the distribution from (b). Assume a_n is a bounded sequence of nonnegative numbers. Show that the "power series with randomized variable"

$$S = \sum_{n=1}^\infty a_n X_n^n$$

converges if and only if $\sum_{n=1}^\infty a_n/n^\alpha < \infty$.

4. Let X and $X_1, X_2, \ldots$ be random variables. Show that $X_n \xrightarrow{d} X$ if and only if $E(g(X_n)) \to E(g(X))$ for all continuous functions $g : \mathbb{R} \to \mathbb{R}$ *with compact support*. (*Hint*. Prove tightness first, and recall that sup can be replaced by lim sup.)

5. Suppose X_n, $n = 1, 2, \ldots$ are i.i.d. with common density

$$f(x) = |x|^{-3} 1_{\{|x| \geq 1\}}.$$

Let $S_n = X_1 + \cdots + X_n$.

(a) Check that $E(X_1) = 0$ but $E(X_1^2) = \infty$.

(b) Prove that $S_n / \sqrt{n \log n} \xrightarrow{d} N(0, 1)$. (*Hint*. One way is to follow these steps: truncate $Y_n = X_n 1_{\{|X_n| \leq \sqrt{n}\}}$, use Lindeberg-Feller to handle the sum of Ys, and L^1 distance to handle the difference between the sum of Xs and the sum of Ys.)

6. Let X_k be independent with $P(X_k = k) = P(X_k = -k) = 1/(2k)$, $P(X_k = 0) = 1 - 1/k$. Let $S_n = X_1 + \cdots + X_n$. Prove that S_n/n converges in distribution to a random variable Z. (*Hint*. Look for a Riemann sum in $\log \varphi_{S_n/n}$.) Show that Z is not Normal, but has a continuous distribution function F_Z.

7. In the *Hat check problem* that was introduced in class, we apply a random permutation π to the set $\{1, 2, ..., n\}$, where each of the possible $n!$ permutations is equally likely. Let X_n be the number of fixed points of the random permutation, that is, $X_n = \sum_{i=1}^n 1_{\{\pi(i)=i\}}$. If Z is a Poisson(1) random variable, show that the first n moments of X_n and Z agree, that is, $E[X_n^k] = EZ^k$ for $k = 1, \ldots, n$. (*Hint*. Argue that this follows from $E[X_n(X_n - 1) \cdots (X_n - k + 1)] = 1$ for $k = 1, 2, \ldots, n$ and devise an inductive argument.)