Math/Stat 235A, Fall 2025

Homework 1, solutions

1. (1) $1-(5/6)^{10}$. (2) $1-2\cdot(5/6)^{10}+(2/3)^{10}$. (3) $\binom{10}{3}\binom{7}{2}6^{-10}$.

2. $\dfrac{1}{6}\cdot\dfrac{1}{\binom{52}{4}}+\dfrac{1}{6}\cdot\dfrac{5}{\binom{52}{4}}+\dfrac{1}{6}\cdot\dfrac{\binom{6}{4}}{\binom{52}{4}}=\dfrac{7}{2\binom{52}{4}}$.

3. $P(E_1)=5p^4(1-p)+p^5=5p^4-4p^5$ and $P(E_2)=1-(1-p)^5-5p(1-p)^4-\binom{4}{2}p^2(1-p)^3-p^3(1-p)^2$. As $E_1\subset E_2$, E_1 and E_2 are not independent.

4. Let $E_1=\{\text{no red toy}\}$, $E_2=\{\text{no blue toy}\}$, $E_3=\{\text{no green toy}\}$. Then $P(E_i)=(1-p/3)^5$, $P(E_i\cap E_j)=(1-2p/3)^5$ for $i\neq j$ and $P(E_1\cap E_2\cap E_3)=(1-p)^5$, so $P(E_1^c\cap E_2^c\cap E_3^c)=1-P(E_1\cup E_2\cup E_3)=1-3(1-p/3)^5+3(1-2p/3)^5-(1-p)^5$.

5.

$$\begin{aligned}P(\text{ticket 1 wins})&=P(\text{two hits on each line})+P(\text{two hits on one line, three on the other})\\&=\frac{3\cdot3\cdot44+2\cdot3}{\binom{50}{5}}=\frac{402}{\binom{50}{5}}\end{aligned}$$

$$\begin{aligned}P(\text{ticket 2 wins})&=P(\text{two hits from 1, 2, 3})+P(\text{three hits from 1, 2, 3})\\&=\frac{3\cdot\binom{47}{3}+\binom{47}{3}}{\binom{50}{5}}=\frac{49726}{\binom{50}{5}}\end{aligned}$$

$$\begin{aligned}P(\text{ticket 3 wins})&=P(\text{2, 3 both hit})+P(\text{1, 4 both hit and one of 2, 3 hit})\\&=\frac{\binom{48}{3}+2\cdot\binom{46}{2}}{\binom{50}{5}}=\frac{19366}{\binom{50}{5}}\end{aligned}$$

$$\begin{aligned}P(\text{ticket 4 wins})&=P(\text{3 hit, at least one additional hit on each line})+P(\text{1, 2, 4, 5 all hit})\\&=\frac{(4\binom{45}{2}+4\cdot45+1)+45}{\binom{50}{5}}=\frac{4186}{\binom{50}{5}}\end{aligned}$$

6. This is the same as choosing at random an $r\times n$ matrix, in which every entry is independently 0 or 1 with probability 1/2, and ending up with at most one 1 in every column. Since columns are independent, this gives $((1+r)2^{-r})^n$

7. Assume that the money unit is \$1,000, call the teams A and B, and assume your client wants to bet on A. Call each sequence of games *terminal* if the series may end with it. To each terminal sequence at which A wins, say AAABA, attach value 2, and to each terminal sequence at which B wins, say BBAAABB, attach value 0. These are the payoffs we need to return. Each non-terminal sequence, say AABA, then has a value which is the average of the two sequences to which it may be extended

by the next game, AABAA and AABAB in this example. This recursively defines the values of all possible sequences. Note that one can extend, with 2's, or respectively 0's, to length 7 all shorter sequences in which A, or respectively B, wins. Then observe that the value of the empty sequence (before games start) is 1: the average of all sequences of length 7 is 1 (by symmetry: the number of sequences with value 0 equals the number of sequences with value 2), and then the average value at each shorter length is also 1. For each sequence s, let $v(s)$ be its value.

Initially, you hold the amount 1. Assume that the games have reached sequence s, you hold the amount of money v, and let sA and sB be the two successors of s. If $v(sB) \leq v(sA)$, bet $v - v(sB)$ on A, else bet $v - v(sA)$ on B. Then its is easy to check by induction that the amount of money you hold at s always equals to the value $v(s)$. (Also note that you do not need to split the penny because the values of sequences of length 1 have at most 2^5 in the denominator and we know that the value is an integer for the sequence of length 0.) Although this might look like a probability problem, it has nothing to do with probability; in particular, the winning probabilities of A and B are irrelevant.