Math/Stat 235A, Fall 2025

Homework 2

Durrett: 1.1.4.

1. (a) For an arbitrary set Ω, assume that $\mathcal{C}$ is a collection of subsets of Ω of cardinality $|\mathcal{C}| = n$. Show that $|\sigma(\mathcal{C})| \le 2^{2^n}$. (*Hint*. Assume first that the events in $\mathcal{C}$ are disjoint and cover Ω.)
(b) Now take a finite probability space Ω, with $|\Omega| = n$, $\mathcal{F} = 2^\Omega$. What is the smallest number of sets which generate $\mathcal{F}$? (In other words, this is the smallest numbers of events one needs to observe to determine the outcome of the experiment.)

2. Let $A_1, A_2, \dots$ be arbitrary events. Denote

$$s_1 = \sum_{i=1}^{\infty} P(A_i),\ s_2 = \sum_{i<j} P(A_i \cap A_j),\ s_3 = \sum_{i<j<k} P(A_i \cap A_j \cap A_k), \dots$$

For (a) and (b) assume that there are only n of these events, i.e., $A_{n+1} = A_{n+2} = \cdots = \emptyset$.

Hint: Although parts (a) and (b) are often proved by induction, it is perhaps clearer to partition Ω into sets $A_1' \cap A_2' \cap \cdots \cap A_n'$ where each A_i' is either A_i or A_i^c. Then count how many times a set in the partition is represented in a probability. The problem them becomes an exercise in combinatorics; in particular, the following identity is useful: $\sum_{i=0}^{k} (-1)^i \binom{n}{i} = (-1)^k \binom{n-1}{k}$, $0 \le k \le n$.
(a) Prove the inclusion–exclusion formula:

$$p = P(A_1 \cup A_2 \cup \cdots \cup A_n) = s_1 - s_2 + s_3 - \cdots + (-1)^{n+1} s_n.$$

(b) Prove the Bonferroni inequalities: $p \le s_1$, $p \ge s_1 - s_2$, $p \le s_1 - s_2 + s_3$, etc.
(c) Which of the following two statements, if any, is true? If $s_1 < \infty$, then $s_2 < \infty$. If $s_2 < \infty$, then $s_1 < \infty$.

3. *A uniform probability measure on* $\mathbb{N}$? For each $A \subset \mathbb{N}$ define

$$\rho_n(A) = \frac{1}{n} \cdot |A \cap [1, n]|$$

and declare that A has *density* $\rho(A)$ if $\rho(A) = \lim_{n\to\infty} \rho_n(A)$ exists. Let $\mathcal{D} = \{A \subset \mathbb{N} : \rho(A) \text{ exists}\}$.
(a) Show that $\mathcal{D}$ is closed under formation of complements, and finite disjoint unions, but not countable disjoint unions or finite non-disjoint unions. Thus $\mathcal{D}$ is not an algebra.
(b) *Optional*. Let $\mathcal{M} = \{a\mathbb{N} : a \in \mathbb{N}\}$ and consider the algebra $a(\mathcal{M})$ generated by $\mathcal{M}$. (*Hint*. Think about an explicit representation of $a(\mathcal{M})$.) Show that $a(\mathcal{M}) \subset \mathcal{D}$, and that ρ is finitely additive, but not countably additive, on $a(\mathcal{M})$.

Hint. For the counterexample to countable additivity, you may wish to follow these steps: (1) Consider the set $B_{L,a} = a\mathbb{N} \setminus (\cup_{p\le L} ap\mathbb{N})$, where p ranges over primes. Justify the use of inclusion-exclusion to compute $\lim_{L\to\infty} \rho(B_{L,a})$. (You will also need the fact that $\sum_p 1/p = \infty$, which you do not need to demonstrate.) (2) Choose L_a so that $B_{L_a,a}$ has a very small density. On the other hand, $\mathbb{N} = \cup_a B_{L_a,a}$.

4. We say that a probability space $(\Omega, \mathcal{F}, P)$ is *non-atomic* if for every measurable set A with $P(A) > 0$ there exists a measurable set $B \subset A$ so that $0 < P(B) < P(A)$.
(a) Show that the standard probability space $([0,1], \mathcal{B}([0,1], P)$, with Lebesgue measure P, is non-atomic. (*Hint*. Show that $f(x) = P(A \cap [0,x])$ is continuous.)
(b) Assume a nonatomic probability space, a mesurable set A with $P(A) > 0$, and an $\epsilon > 0$. Show that there exists a measurable set $B \subset A$ with $0 < P(B) < \epsilon$.
(c) *Optional*. Assume a nonatomic probability space, a mesurable set A with $P(A) > 0$, and an $\alpha \in [0, P(A)]$. Show that there exists a measurable set $B \subset A$ with $P(B) = \alpha$.

Here is the argument using the axiom of choice (adapted from the *Stack Exchange*). Recall the definition of symmetric difference of two sets: $E_1 \triangle E_2 = (E_2 \setminus E_1) \cup (E_1 \setminus E_2)$. Define $\mathcal{P} = \{E \in \mathcal{F} : E \subset A, P(E) \leq \alpha\}$. Define the equivalence relation on $\mathcal{P}$ by $E_1 \sim E_2$ if $P(E_1 \triangle E_2) = 0$, and let $[\mathcal{P}]$ be the set of resulting equivalence classes. Make $[\mathcal{P}]$ a partially ordered set by this ordering relation on $\mathcal{P}$: $E \leq F$ if $P(E \setminus F) = 0$. (*Note*: the equivalence classes are needed for antisymmetry; the probability P is constant on any equivalence class; and the order on $\mathcal{P}$ lifts to $[\mathcal{P}]$ as $E_1 \setminus F_1 \subset (E_2 \setminus F_2) \cup (E_1 \triangle E_2) \cup (F_1 \triangle F_2)$ and $E_2 \setminus F_2 \subset (E_1 \setminus F_1) \cup (E_1 \triangle E_2) \cup (F_1 \triangle F_2)$.) We first claim that every chain $\mathcal{C} \subset [\mathcal{P}]$ has an upper bound. To prove this claim, let $\beta = \sup_{[E] \in \mathcal{C}} P(E)$, and let $[E_1], [E_2], \ldots \in \mathcal{C}$ be a countable sequence such that $P(E_n) \uparrow \beta$. Then the equivalence class of $E = \bigcup_{n=1}^{\infty} E_n$ is the desired upper bound. Indeed, given any $[F] \in \mathcal{C}$, there are two cases. If for some n, we have $E_n \geq F$, then clearly $E \geq F$. Otherwise, $E_n \leq F$ for all n, which implies $P(F) = \beta$ and

$$P(F \setminus E) = \lim_n P(F \setminus E_n) = \lim_n (P(F) - P(E_n)) = \beta - \beta = 0,$$

so again, $F \leq E$. As every chain has an upper bound, Zorn's Lemma implies that $[\mathcal{P}]$ has a maximal element, say $[G]$. We claim that $P(G) = \alpha$. If not, then $\alpha - P(G) > 0$, so using (b) we can find a measurable set $H \subset A \setminus G$ with $0 < P(H) < \alpha - P(G)$, so that $G \cup H \in \mathcal{P}$, with $P(G \cup H) > P(G)$, contradicting the maximality of $[G]$.