Math/Stat 235A, Fall 2025

Homework 2, Solution sketches.

Durrett: 1.1.4. Part (i) is routine.

For (ii), take $\Omega = \mathbb{N} = \{1, 2, \dots\}$ and $\mathcal{F}_n$ to be the σ-algebra generated by all subsets of $I_n = \{1, \dots, n\} \subset \mathbb{N}$. Then check that $\mathcal{F}_n$ comprises exactly all sets of the form $A \cup B$, where A is any subset of I_n and B is either $\emptyset$ or $\mathbb{N} \setminus I_n$. These are clearly increasing σ-algebras. The union $\cup_n \mathcal{F}_n$ does not include any subset of $\mathbb{N}$ which is infinite and has infinite complement, for example, the set E of even numbers. However, $\{k\} \in \mathcal{F}_k \subset \cup_n \mathcal{F}_n$ for all $k \in \mathbb{N}$, and $E = \cup_{k \in E}\{k\}$.

1. (a) Form 2^n disjoint sets $B_J = A_1' \cap \cdots \cap A_n'$, where $J \subset \{1, \dots, n\}$ and A_i' equals A_i if $i \in J$ and A_i^c otherwise. Then $\sigma(\{A_i\}) = \sigma(\{B_J\})$, since A_i is the union over all B_J of the form $A_1' \cap \cdots \cap A_{i-1}' \cap A_i \cap A_{i+1}' \cap \cdots \cap A_n'$. Also, B_J are disjoint, and the σ–algebra generated by B_J containts exactly all sets of the form

$$\bigcup_J B_J,$$

where J ranges over any selection of different subsets of $\{1, \dots, n\}$.
(b) From (a), we know that the smallest number of such sets is at least $\lceil \log_2 n \rceil$. To prove that a generating set with such cardinality exists, we can restrict to powers of 2, say $n = 2^k$, and assume $\Omega = \{1, \dots, n\}$. We will then prove this by induction on k. For $k = 1$, the set $\{1\}$ clearly generates. Given $n = 2^{k+1}$, list pairs $1-2$, $3-4$, etc., and form the k sets from the inductively given solution for $n = 2^k$ in which each element i is replaced by the i'th pair. The $(k+1)$'st set, $\{2, 4, \dots, 2^{k+1}\}$ takes one element from each pair. By the inductive hypothesis, the σ-algebra includes all unions of pairs; the intersection with the last set gives all subsets of even numbers, then the intersection with complement of the last set includes all subsets of odd numbers, and finally unions of these two types give all subsets.

2. (a) and (b). Consider all the sets in the hint, excluding the one with $A_i' = A_i^c$ for all i. Each of these sets is represented once in the left hand side. For the right hand side, consider, say, the set, $A_1 \cap \cdots \cap A_k \cap A_{k+1}^c \cap \cdots \cap A_n^c$, for some $k \geq 1$. This set is only contained in an intersection with indices $\leq k$, and, if the number of indices is ℓ, the set is represented $\binom{k}{\ell}$ times. The desired conclusions now follow from the combinatorial formula.
(c) We have, by considering finite unions and taking limits, $s_1 \leq P(\cup_i A_i) + s_2 \leq 1 + s_2$, so the second implication holds. A counterexample to the first one are *decreasing* events A_i with $P(A_i) = 1/i^2$. Then $s_1 < \infty$, while $s_2 = \sum_{j=2}^{\infty}(j-1)/j^2 = \infty$.

3. (a) Here is a set without a density: $A_1 = \cup_{k \geq 0, k \text{ even}}[2^k, 2^{k+1})$, which demonstrates that $\mathcal{D}$ is not closed under formation of countable disjoint unions. Also, take a set A_2 consisting of even integers in A_1 and odd ones in A_1^c. Moreover, let $A_3 = 2\mathbb{N}$. Then $\rho(A_2) = \rho(A_3) = 1/2$ but $A_2 \cap A_3$ does not have a density.
(b) For an explicit representation of $a(\mathcal{M})$, pick a positive integer n and pick sets $A_1, \dots, A_n \in \mathcal{M}$. For a set $J \subset \{1, \dots, n\}$, let B_J be as in 1(a). Let $\mathcal{J}$ be any selection of subsets of $\{1, \dots, n\}$. Then

$$a(\mathcal{M}) = \{\bigcup_{J \in \mathcal{J}} B_J : n, \mathcal{J}\}.$$

(The straightforward verification that the set on the right forms an algebra is omitted.) To show that

$a(\mathcal{M}) \subset \mathcal{D}$, note that the union above is disjoint, so we need to show that any intersection that forms B_J above is in $\mathcal{D}$. That is, we need to show that

$$(a\mathbb{N}) \cap (b_1\mathbb{N})^c \cap \cdots \cap (b_n\mathbb{N})^c \in \mathcal{D}$$

for any n and any choice $a, b_1, \ldots, b_n \in \mathbb{N}$. We will show this by induction on n. For $n = 1$ note that

$$(a\mathbb{N}) \cap (b\mathbb{N})^c = (a\mathbb{N}) \cap (\mathrm{lcm}(a, b)\mathbb{N})^c = ((a\mathbb{N})^c \cup (\mathrm{lcm}(a, b)\mathbb{N}))^c \in \mathcal{D}$$

(lcm=least common multiple), because $\mathcal{D}$ is closed under disjoint finite unions. For $n-1 \to n$ step,

$$\begin{aligned}&(a\mathbb{N}) \cap (b_1\mathbb{N})^c \cap \cdots \cap (b_n\mathbb{N})^c \\ &= [(a\mathbb{N}) \cap (b_1\mathbb{N})^c \cap \cdots \cap (b_{n-1}\mathbb{N})^c] \cap [(a\mathbb{N}) \cap (b_1\mathbb{N})^c \cap \cdots \cap (b_{n-1}\mathbb{N})^c \cap (b_n\mathbb{N})]^c .\end{aligned}$$

As above, it is enough to show that both sets inside the square brackets are in $\mathcal{D}$. This follows from the induction hypothesis (note that $(a\mathbb{N}) \cap (b_n\mathbb{N}) = \mathrm{lcm}(a, b_n)\mathbb{N}$).

Finite additivity on $a(\mathcal{M})$ now follows, since ρ is finitely additive on $\mathcal{D}$. Now inclusion–exclusion formula only needs finite additivity, so

$$\rho(B_{L,a}) = \frac{1}{a} - \sum_{p \le L} \frac{1}{ap} + \sum_{p_1 < p_2 \le L} \frac{1}{ap_1p_2} - \cdots = \frac{1}{a} \cdot \prod_{p \le L} \left(1 - \frac{1}{p}\right).$$

This goes to 0 (for arbitrary a) as $L \to \infty$. So choose L_a so that $\rho(B_{L_a,a}) < 2^{-a}$. Since $a \in B_{a,L_a}$, countable additivity would imply

$$\rho(\mathbb{N}) = \rho(\cup_{a=1}^{\infty} B_{a,L_a}) \le \sum_{a=1}^{\infty} \rho(B_{a,L_a}) < \sum_{a=1}^{\infty} 2^{-a} = 1,$$

a contradiction.

4. (a) For continuity from the left, say, assume $x_n \uparrow x$, then $A \cap [0, x_n] \uparrow A \cap [0, x)$ and so $f(x_n) = P(A \cap [0, x_n]) \to P(A \cap [0, x)) = P(A \cap [0, x]) = f(x)$. Once you establish continuity, use the intermediate value thorem.
(b) If $B \subset A$ such that $0 < P(B) < P(A)$, then either $0 < P(B) \le \frac{1}{2}P(A)$ or $0 < P(B \setminus A) < \frac{1}{2}P(A)$. Iterate this process with A repleaced by either B or $A \setminus B$.