Math 235A, Fall 2025

Homework 3

Durrett: 1.2.4. (*Hint*. Consider the "inverse" $\varphi(y) = \sup\{z : F(z) < y\}$ that we defined in class. Show that $F(\varphi(y)) = y$.)

1.(a) Assume that $\mathcal{M}$ is a set of measurable functions on $(\Omega, \mathcal{F}, P)$, $\mathcal{A} \subset \mathcal{F}$ is a π-system and:

(i) If $f_1, f_2 \in \mathcal{M}$, $a, b \in \mathbb{R}$, then $af_1 + bf_2 \in \mathcal{M}$.

(ii) If $f_1 \le f_2 \le f_3 \le \dots$ is a sequence of functions in $\mathcal{M}$ and $f(\omega) = \lim_{n\to\infty} f_n(\omega) < \infty$ for each $\omega \in \Omega$, then $f \in \mathcal{M}$.

(iii) $1_A \in \mathcal{M}$ for each $A \in \mathcal{A}$.

Then $\mathcal{M}$ contains all $\sigma(\mathcal{A})$-measurable functions. Prove this (so–called *Monotone class theorem*).

(*Hint*. First use π-λ theorem to conclude that (iii) holds for each $A \in \sigma(\mathcal{A})$. Then use approximation with simple functions.)

(b) Let X, Y be random variables on $(\Omega, \mathcal{F}, P)$. Assume that Y is $\sigma(X)$-measurable. Show that there is a Borel function $f : \mathbb{R} \to \mathbb{R}$ so that $Y = f(X)$.

(*Hint*. Define $\mathcal{M}$ to be the class of all random variables of the form $f(X)$ where f is a Borel function, and $\mathcal{A} = \sigma(X)$. Show that $\mathcal{M}$ satisfies (i), (ii), (iii) above.)

(c) Let X, Y be random variables on $(\Omega, \mathcal{F}, P)$. If Y is $\sigma(X)$-measurable, must X be $\sigma(Y)$-measurable?

2. From *New York Times*, April 10, 2001:

> "Three players enter a room and a red or blue hat is placed on each person's head. The color of each hat is determined by [an independent] coin toss. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats [but not their own], the players must simultaneously guess the color of their own hats or pass. The puzzle is to find a group strategy that maximizes the probability that at least one person guesses correctly and no-one guesses incorrectly."

One strategy would be for the group to agree that one person should guess and the others pass. This would have probability 1/2 of success. Find a strategy with a greater chance for success.

Now generalize the problem by allowing every person i to place an even bet x_i on the color of his or her own hat. The bet can either be on red or on blue and the amount x_i of the bet is arbitrary. (*Even bet* means a correct guess wins x_i, while an incorrect guess means loss of the same amount.) The group wins if their combined wins are strictly greater than their losses. Find, with proof, a strategy with maximal winning probability. Generalize to n people.

3. Assume that μ is a probability measure on $\mathcal{B}(\mathbb{R}^3)$. A plane $\pi \subset \mathbb{R}^3$ is a *fair cut* (for μ) if μ of both components of π^c is exactly 1/2. Prove the following three statements.

(a) If $\mu(\pi) = 0$ for every plane π, there is a fair cut perpendicular to the z-axis.

(b) *Optional*. If $\mu(\ell) = 0$ for every line ℓ, there is a fair cut through the origin. (*Hint*. The number of different planes π for which $\mu(\pi) > 0$ is at most countable.)

(c) *Optional*. If $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}^3$, there is a fair cut.