Math 235A, Fall 2025

Homework 3, Solution sketches.

Durrett, 1.2.4. Fix a $y \in (0,1)$. Let $\varphi(y) = \sup\{z : F(z) < y\}$. We proved in class that $\varphi(y) \le x$ if and only if $y \le F(x)$. This is true for all F.

We claim first that for all y, $F(\varphi(y)) = y$. If $F(w) > y$, then by right-continuity of F, $F(w-\epsilon) > y$ for some $\epsilon > 0$, and so $\varphi(y) < w$. It follows that $F(\varphi(y)) \le y$. If $F(w) < y$, then by the assumed left-continuity of F, $F(w+\epsilon) < y$ for some $\epsilon > 0$, and so $\varphi(y) > w$. It follows that $F(\varphi(y)) \ge y$. This establishes the claim.

Note that, by continuity, $P(X \ge x) = 1 - F(x)$, for all x. It now follows that

$$P(Y \ge y) = P(F(X) \ge y) = P(X \ge \varphi(y)) = 1 - F(\varphi(y)) = 1 - y,$$

and so $P(Y < y) = y$, for all y, but then also $P(Y \le y) = \lim_{z\downarrow y} P(Y < z) = \lim_{z\downarrow y} z = y$ for all y.

1. (a) $A \subset B$ implies $1_{B\setminus A} = 1_B - 1_A$ and $A_n \uparrow A$ implies $1_{A_n}(\omega) \uparrow 1_A(\omega)$ for each ω. Hence the set of all A for which $1_A \in \mathcal{M}$ is a λ-system (by (i) and (ii)), which contains $\mathcal{A}$ (by (iii)) hence by the $\pi - \lambda$ theorem (iii) holds for each $A \in \sigma(\mathcal{A})$. Using (i) again, one can conclude that each simple $\sigma(\mathcal{A})$-measurable function is in $\mathcal{M}$. If now $X \ge 0$ is $\sigma(\mathcal{A})$-measurable function, then there is a sequence of $\sigma(X)$-measurable (hence $\sigma(\mathcal{A})$-measurable) simple functions X_n which increases to X. By (ii), $X \in \mathcal{M}$. Finally, if X is any $\sigma(\mathcal{A})$ measurable function, write $X = X_+ - X_-$.
(b) If $Y_1 = f_1(X)$ and $Y_2 = f_2(X)$ are in $\mathcal{M}$ then $aY_1 + bY_2 = (af_1 + bf_2)(X) \in \mathcal{M}$. If $Y_1 = f_1(X) \le Y_2 = f_2(X) \le \dots$ are all in $\mathcal{M}$, then $Y = \lim Y_n = f(X)$ where $f(x) = \lim f_n(x)$ if the limit exists (and is finite) and $f(x) = 0$ otherwise. Note that the limit must exist for any x in the range of X. Finally, for each $A \in \mathcal{A}$ (which clearly is a π-system) there exists a Borel set B such that $A = \{X \in B\}$, hence $1_A = 1_B(X)$.
(c) No. For example, Y may be a constant, while X is not.

2. The first question is a classic puzzle. If you see same colors, guess the color you do not see. If you see different colors, pass. The probability of a win is then 3/4. (Maximizing the probability in this context is a difficult problem for a large number of people.)

For the second question, call the two colors $+$ and $-$ for red and blue, label people $1, \dots, n$ and imagine them ordered on a line. Every possible strategy can be described as n functions F_i, $i = 1, \dots, n$, $F_i : \{+,-\}^{n-1} \to \mathbb{R}$, which could be interpreted as i's bet on $+$ provided i sees the configuration of signs in the given order. (The negative values of F are bets on $-$.) For example, the payoff at configuration $+--$ (for $n = 3$) then is $F_1(--) - F_2(+-) - F_3(+-)$. There are 2^n configurations, hence these many payoffs. We need to make as many of these positive as possible. On the other hand, to specify a strategy we need to specify $n \cdot 2^{n-1}$ bets. This makes it look like all payoffs can be made positive, but this is not so. Denote by x a generic n–configuration and x^i the $(n-1)$–configuration obtained by removing the i'th coordinate from x. Then the expected payoff is

$$\frac{1}{2^n} \sum_x \sum_{i=1}^n \pm F_i(x^i) = 0,$$

as every $F_i(y)$ appears in the sum twice, with different signs. As a consequence, at most $2^n - 1$ payoffs can be made positive.

To show that this is indeed possible, define F_i as follows. Pick $y \in \{+,-\}^{n-1}$. If there is a $+$ among first i coordinates of y, then $F_i(y) = 0$, otherwise $F_i(y) = 2^{i-1}$. (Thus, the 1'st person places a unit bet, $F_1 \equiv 1$, and the i'th person bets 2^{i-1} if there are no red hats in front of him, and otherwise abstains from betting.) If there is at least one red hat, the winning amount of the first person with a red hat is larger than the combined losses of the people before him (as $1 + \ldots + 2^{i-2} < 2^{i-1}$ for $i \geq 2$), and the remaining people do not bet. The only losing configuration is the one with no red hat, when everybody bets and loses.

3. (a) Take the half-space $H_a = \{(x, y, z) : z < a\}$. Then $\mu(H_a)$ goes to 0 as $a \to -\infty$, goes to 1 as $a \to \infty$ and is continuous (since μ of any plane $\{z = a\}$ is 0). Hence $\mu(H_a) = 1/2$ for some a.
(b) To prove the claim in the hint, we fix a positive integer n, and show that there are at most n different planes with measure μ larger than $1/n$. Indeed, if there were $n+1$ such planes $\pi_1, \ldots, \pi_{n+1}$, then, by inclusion-exclusion

$$\mu(\cup_{i=1}^{n+1}\pi_i) = \sum_{i=1}^{n+1}\mu(\pi_i) - \sum_{1\leq i<j\leq n+1}\mu(\pi_i \cap \pi_j) + \sum_{1\leq i<j<k\leq n+1}\mu(\pi_i \cap \pi_j \cap \pi_k) - \cdots = \sum_{i=1}^{n+1}\mu(\pi_i),$$

since the intersection of two or more different planes is a line, a point or empty. But the last sum is at least $(n+1)/n$, a contradiction.

A plane through the origin can be identified with its unit normal, that is, a vector $u \in S^2$, where S^2 the unit sphere in $\mathbb{R}^3$. Call these planes π_u (the map $u \mapsto \pi_u$ is a 2-to-1 map). By the above paragraph, there must exist a great circle $C \subset S^2$ such that $\mu(\pi_u) = 0$ for every $u \in C$. By continuity again, one of these π_u must be a fair cut.
(c) As above, one can prove that a set of lines ℓ with $\mu(\ell) > 0$ is countable. Also countable is the set of planes π such that $\mu(\pi) > 0$ and $\mu(\ell) = 0$ for every line $\ell \subset \pi$, by the argument in (b). Call such lines and planes *exceptional*.

Let now $E \subset S^2$ be the set of all vectors u such that none of the planes with normal u contains an exceptional line. Since every one of exceptional lines only eliminates one great circle from S^2, and there are countably many exceptional lines, the Lebesgue measure (i.e., the surface measure on S^2) of E is 1. Hence there must also exist an $u \in E$ such that no plane with normal u is exceptional. Therefore, all planes with normal u have probability 0, and the argument in (a) finishes the proof.