Math 235A, Fall 2025

Homework 4

Durrett: 1.6.6, 1.6.14.

1. (a) Let U be a random variable, uniformly distributed on $[0,1]$. Define $X_i, i = 1, 2, \ldots$ to be the i'th digit in the binary expansion of U. Show that X_i are i.i.d. random variables.
(b) Assume you have a sequence of i.i.d. random variables $X_i, i = 1, 2, \ldots$ on a probability space $(\Omega, \mathcal{F}, P)$ with $P(X_i = 1) = P(X_i = 0) = 1/2$. Given a distribution function F, use X_i to construct (on the same probability space) a sequence of i.i.d. random variables $Y_1, Y_2, \ldots$ with distribution function F. (*Hint*. Show first that $U = \sum_{i=1}^{\infty} X_i/2^i$ is a uniform random variable. Then use U to find *one* random variable with distribution function F.)

2. Let A be a random $n \times n$ matrix, whose entries X_{ij} are independent and $P(X_{ij} = 1) = P(X_{ij} = -1) = 1/2$. Compute $\mathrm{Var}(\det(A))$.

3. Somebody chooses two real random variables X and Y and writes them on two sheets of paper. The distrubution of (X, Y) is unknown to you, but you do know that $P(X = Y) = 0$. Based on a toss of a fair coin, independent of (X, Y), you choose one of the sheets and observe the number on it. Call this random number W and the other number, still unknown to you, Z. Your task is to guess whether W is bigger than Z or not. Assume you can generate independent uniform (on $[0,1]$) random variables at will, so your strategy could be random.
(a) Show that, if your guess is always that $W > Z$, then the probability of being correct is exactly $1/2$.
(b) Exhibit a stategy for which the probability of being correct is $1/2 + \epsilon$, for some $\epsilon > 0$. This ϵ may depend on the distribution of (X, Y), but your strategy of course can not. (*Hint*. Take a random variable G with positive density everywhere and independent of everything else, and guess $W > Z$ iff $W > G$.)
(c) Assume that you know that X and Y will be chosen from among integers in $[0, 10]$. Find, with proof, the largest ϵ which works for all such distributions.