Math 235A, Fall 2025

Homework 4, Solution sketches.

Durrett, 1.6.6. We have $(EY)^2 = E(Y \cdot 1_{\{Y>0\}})^2 \le E(Y^2)E(1_{\{Y>0\}}) = E(Y^2)P(Y > 0)$.

Durrett, 1.6.14. The expression inside the two limits is $E(\frac{y}{X} \cdot 1_{\{X>y\}})$. Let $Z_y = \frac{y}{X} \cdot 1_{\{X>y\}}$. Then $Z_y \le 1$, so one can apply the dominant convergence theorem. But, $Z_y \to 0$ as either $y \to \infty$ or $y \to 0$.

1. (a) Let $x_1, \dots, x_n \in \{0, 1\}$, and $a = 0.x_1x_2...x_n$. Then $P(X_1 = x_1, \dots, X_n = x_n) = P(U \in [a, a + 2^{-n})) = 2^{-n}$, which shows that X_i are independent with $P(X_i = x_i) = 1/2$.
(b) Pick $x \in (0, 1)$ with $0.x_1x_2...$ being its binary expansion. (To be specific, we use, say, only binary expansions that have infinitely many 0's.) Let N be the first n so that $X_n \ne x_n$. Then $P(U < x) = \sum_{n=1}^{\infty} P(N = n, X_n = 0, x_n = 1) = \sum_{n=1}^{\infty} x_n/2^n = x$, hence U is uniform. Then let $\phi : (0, 1) \to \mathbf{R}$ be the function $\phi(u) = \sup\{y : F(y) < u\}$. We showed in class that $\phi(U)$ has d.f. F. To find i.i.d. sequence with d.f. F, let, say, $p_k, k = 1, 2, \dots$ be the sequence of primes and let $Y_k = \phi(\sum_{n=1}^{\infty} X_{2^n p_k}/2^n)$.

2. We use

$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi) \prod_{i=1}^{n} X_{i,\pi(i)},$$

where π is a generic permutation of $1, \dots, n$. The random variables inside the sum have expectation 0, are uncorrelated, and have values ± 1. As the sum has $n!$ terms, $\operatorname{Var}(\det(A)) = n!$.

3. (a)

$$\begin{aligned} P(W > Z) &= P(W = X, X > Y) + P(W = Y, X < Y) \\ &= (1/2)P(X > Y) + (1/2)P(X < Y) = 1/2. \end{aligned}$$

(b) Let G be an $N(0, 1)$ random variable (or any other random variable with density which is positive everywhere on $\mathbb{R}$), independent of X, Y and the toss of the coin. The strategy is to guess that $W > Z$ if $W > G$ and that $W < Z$ if $W < G$. In this case,

$$\begin{aligned} &P(\text{correct guess}) \\ &= P(W > Z, W > G) + P(W < Z, W < G) \\ &= (1/2)[P(X > Y, X > G) + P(Y > X, Y > G) + P(X < Y, X < G) + P(Y < X, Y < G)] \\ &= (1/2)[P(X > Y) + P(X > Y, Y < G < X) + P(X < Y) + P(X < Y, X < G < Y)] \\ &= 1/2 + (1/2)P(G \text{ between } X \text{ and } Y) > 1/2. \end{aligned}$$

(c) All possible strategies are given by p_k = probability that you guess W is smaller, provided W turns out to be k, $k = 0, 1, 2, \dots, 10$. Assume that X and Y are chosen deterministically: $X = n$, $Y = n+1$, for some $n = 0, \dots, 9$. The probability of you guessing correctly then is $1/2+(p_n-p_{n+1})/2$. Therefore,

$$\epsilon \le \min_n (p_n - p_{n+1})/2 \le 0.05,$$

as among 10 numbers whose sum is at most 1 not all can exceed 0.1. To show that the above ϵ can be achieved, take let G in (b) be a uniform on $[0, 10]$ (or, equivalently, $p_k = k/10$), to get $P(G$ between X and $Y) \ge 0.1$. This shows that the probability of a correct guess is at least 0.55.