Math 235A, Fall 2025

Homework 5

Durrett: 2.2.8 (Demonstrate the conclusion by directly verifying that the triangular array theorem holds with $b_n = n/\log_2 n$ rather that the way suggested in the book.)

1. Let X_n, $n = 1, 2, \ldots$, be a sequence of r.v.'s defined on a common probability space. Show carefully that the following statements are equivalent.
(1) $X_n \to 0$ a.s.
(2) For every $\epsilon > 0$, $P(|X_n| > \epsilon \text{ i.o.}) = 0$.
(3) For every $\epsilon > 0$ and every sequence of integer valued random variables $N_k \geq 1$ such that $N_k \to \infty$ in probability, $P(|X_{N_k}| > \epsilon) \to 0$ as $k \to \infty$.

Remarks. To formally define what it means that $N_k \to \infty$ in probability: for every $\epsilon > 0$, $P(N_k \geq 1/\epsilon) \to 1$ as $k \to \infty$. As N_k are positive, this is equivalent to $1/N_k \to 0$ in probability. Also, it is a good exercise to formally prove that X_{N_k} are indeed random variables.

2. A light bulb goes out in any single day (independently of other days) with probability $p \in (0, 1)$, and after it goes out it remains out. Suppose there are n bulbs at the beginning of day 1, that they behave independently, and let T be the number of days it takes for all the bulbs to go out. Compute $E(T)$ exactly (for every p and n). Then think of p as fixed and let $n \to \infty$. Determine the asymptotic behavior of $E(T)$ and show that $T/E(T)$ converges to 1 in probability.