Homework 5, Solution sketches

Durrett, 2.2.8. We will verify the two conditions of the theorem with $b_n = n/\log_2 n$. To verify (i),

$$nP(X_1 > b_n) = n \sum_{k > \log_2(b_n+1)} \frac{1}{2^k k(k+1)} \asymp \frac{n}{b_n \log_2^2 b_n} \asymp \frac{1}{\log_2 n}.$$

Here, we use $\asymp$ to indicate that the ratio of the two quantities is aysmptotically between two constants. To verify (ii),

$$\frac{n}{b_n^2} E\left(X_1^2 1_{\{X_1 \le b_n\}}\right) \le \frac{n}{b_n^2} \sum_{k \le \log_2(b_n+1)} \frac{2^k}{k(k+1)} \asymp \frac{n b_n}{b_n^2 \log_2^2 b_n} \asymp \frac{1}{\log_2 n}.$$

One way to establish the asymptotics of sums as the one above is to use the discrete L'Hôpital (known as Stolz Theorem or Stolz-Cesàro Theorem). Finally,

$$\begin{aligned} E\left(X_1 1_{\{|X_1| \le b_n\}}\right) &= -E\left(X_1 1_{\{X_1 > b_n\}}\right) \\ &= -\sum_{k > \log_2(b_n+1)} \frac{1}{k(k+1)} + \sum_{k > \log_2(b_n+1)} \frac{1}{2^k k(k+1)} \\ &= -\frac{1}{\log_2 b_n} + o\left(\frac{1}{\log_2 b_n}\right). \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{-n}{b_n \log b_n} = -1.$$

1.
(1) $\implies$ (2): Clear.
(2) $\implies$ (1): Let $\epsilon_k > 0$ be a sequence of numbers decreasing to 0. Than, for every k, $P(\{|X_n| > \epsilon_k$ i.o.$\}) = 0$. Then $P(\cup_k \{|X_n| > \epsilon_k$ i.o.$\}) = 0$ and so $P(\cap_k \{|X_n| \le \epsilon_k$ ev.$\}) = 1$. But $\cap_k \{|X_n| \le \epsilon_k$ ev.$\} \subset \{\lim X_k = 0\}$.
(2) $\implies$ (3): For any fixed n, $P(N_k \le n) \to 0$ as $k \to \infty$. Therefore,

$$\begin{aligned} &P(|X_{N_k}| > \epsilon) \\ &= P(|X_{N_k}| > \epsilon, N_k \le n) + P(|X_{N_k}| > \epsilon, N_k > n) \\ &\le P(N_k \le n) + P\left(\cup_{m \ge n} \{|X_m| > \epsilon\}\right). \end{aligned}$$

Therefore, for all n,

$$\limsup_{k\to\infty} P(|X_{N_k}| > \epsilon) \le P\left(\cup_{m\ge n}\{|X_m| > \epsilon\}\right),$$

and then sending $n \to \infty$,

$$\limsup_{k\to\infty} P(|X_{N_k}| > \epsilon) \le P\left(\cap_n \cup_{m \ge n} \{|X_m| > \epsilon\}\right) = P(|X_n| > \epsilon \text{ i.o.}) = 0.$$

(3) $\implies$ (2): Assume that (2) is not true, that is, that there exist $\epsilon > 0$ and $\alpha > 0$ so that $P(|X_n| > \epsilon$ i.o.$) \ge \alpha$. Define a sequence N_k inductively: $N_1 = 1$ and N_{k+1} is the smallest $m > N_k$ for

which $|X_m| > \epsilon$. If such a m does not exist, simply define $N_{k+1} = N_k + 1$. Then $N_k \to \infty$ for every ω and, on the event $\{|X_n| > \epsilon \text{ i.o.}\}$, $|X_{N_k}| > \epsilon$ for all $k \geq 2$. It follows that $P(|X_{N_k}| > \epsilon) \geq \alpha$ for all $k \geq 2$, so that (3) does not hold.

To prove that X_{N_k} are indeed random variables, observe that, for $B \in \mathcal{B}(\mathbb{R})$

$$\{X_{N_k} \in B\} = \bigcup_{n \geq 1} \left(\{X_n \in B\} \cap \{N_k = n\}\right),$$

and the event inside the union is in $\mathcal{F}$ for all n as both X_n and N_k are random variables.

2. Let $q = 1 - p$. As $P(T > k) = 1 - (1 - q^k)^n$,

$$E(T) = \sum_{k=0}^{\infty} \left[1 - (1 - q^k)^n\right].$$

We will use the fact that there exists a constant $c > 0$ (depending on q) so that for $x \in [0, q]$, $e^{-cx} \leq 1 - x \leq e^{-x}$. Therefore, for $k \geq 1$,

$$P(T > k) \leq 1 - e^{-cnq^k} \leq cnq^k$$

and

$$P(T > k) \geq 1 - e^{-nq^k}.$$

Let $a_n = -\log n / \log q$. The above inequalities show that $T/a_n \to 1$ in probability. Also, for any $\epsilon > 0$,

$$\sum_{k \geq (1+\epsilon)a_n} P(T > k) \leq c\frac{1}{1-q} \cdot nq^{(1+\epsilon)a_n} = c\frac{1}{1-q} \cdot n^{-\epsilon}$$

and

$$\sum_{k \leq (1-\epsilon)a_n} P(T > k) \geq (1-\epsilon)a_n \left(1 - e^{-n^\epsilon}\right).$$

Therefore,

$$(1-\epsilon)\left(1 - e^{-n^\epsilon}\right) \leq a_n^{-1} E(T) \leq (1+\epsilon) + a_n^{-1} + ca_n^{-1}\frac{1}{1-q} \cdot n^{-\epsilon}.$$

This proves $E(T)/a_n \to 1$.