Homework 6, Solution sketches

Durrett, 2.4.1. Let $A_n = \sum_{k=1}^n X_i$, $B_n = \sum_{k=1}^n Y_i$, and $T_n = A_n + B_n = \sum_{k=1}^n (X_i + Y_i)$. Also let $N_t = \sup\{n : T_n < t\}$ be the number of light bulb changes in $[0,t)$. Then, $T_{N_t} < t \le T_{N_t+1}$. Clearly $N_t \to \infty$ as $t \to \infty$ a.s. Moreover,

$$\frac{R_t}{t} \le \frac{A_{N_t} + X_{N_t+1}}{t} \le \frac{A_{N_t} + X_{N_t+1}}{T_{N_t}} = \frac{A_{N_t}/N_t}{T_{N_t}/N_t} + \frac{X_{N_t+1}}{N_t} \cdot \frac{1}{T_{N_t}/N_t}$$

and

$$\frac{X_{N_t+1}}{N_t} = \frac{A_{N_t+1} - A_{N_t}}{N_t} = \frac{A_{N_t+1}}{N_t+1} \cdot \frac{N_t+1}{N_t} - \frac{A_{N_t}}{N_t}.$$

Let Ω_0 be the set of outcomes for which $N_t \to \infty$ as $t \to \infty$, and $A_n/n \to EX_1$, $B_n/n \to EY_1$, as $n \to \infty$. On Ω_0, $A_{N_t}/N_t \to EX_1$, $A_{N_t+1}/(N_t+1) \to EX_1$, $T_{N_t}/N_t \to EX_1 + EY_1$, and then, by the above computations,

$$\limsup \frac{R_t}{t} \le \frac{EX_1}{EX_1 + EY_1}.$$

To get the lower bound,

$$\frac{R_t}{t} \ge \frac{A_{N_t}}{t} \ge \frac{A_{N_t}}{T_{N_t+1}} = \frac{A_{N_t}/N_t}{T_{N_t+1}/(N_t+1)} \cdot \frac{N_t}{N_t+1},$$

which gives

$$\liminf \frac{R_t}{t} \ge \frac{EX_1}{EX_1 + EY_1}.$$

Durrett, 2.4.2. Let $R_1, R_2, \dots$ be i.i.d., with $R_i \overset{d}{=} |Z|$, where Z is distributed uniformly in the unit circle. Then $|X_n| = R_1 R_2 \cdots R_n$ and so the claim follows by SLLN, with

$$c = E(\log R_1) = E(\log |Z|) = 2\int_0^1 r \log r\, dr = \left(r^2 \log r - \frac{1}{2} r^2\right)\Big|_{r=0}^{r=1} = -1/2.$$

Durrett, 3.2.13. Fix an $\epsilon > 0$. Then $P(X_n + Y_n \le x) \le P(X_n + c \le x + \epsilon) + P(|Y_n - c| \ge \epsilon)$ and $P(X_n + Y_n \le x) + P(|Y_n - c| \ge \epsilon) \ge P(X_n + c \le x - \epsilon)$. If x is a continuity point of the distribution function of $X + c$ (that is, $P(X + c = x) = 0$), send $n \to \infty$, then send $\epsilon \to 0$.

Durrett, 3.2.14. Again, fix $\epsilon > 0$. Assume $x > 0$. Then

$$P(X_n Y_n \le x) \le P(X_n \cdot c \le x(1+\epsilon)) + P(Y_n \le c/(1+\epsilon)),$$
$$P(X_n Y_n \le x) \ge P(X_n \cdot c \le x(1-\epsilon)) - P(Y_n \ge c/(1-\epsilon)),$$

and proceed as above. If $x < 0$, replace X_n by $-X_n$ and x by $-x$ and proceed similarly.

Durrett, 3.2.15. As X_n, given in the hint, is invariant under orthogonal transformations, X_n is uniform on S^n. By SLLN, $\sqrt{n/\sum_{m=1}^n Y_m^2}$ converges to 1 almost surely. So, X_n^1 converges to Y_1 almost surely, hence in distribution.

1. The a. s. convergence to 1 follows because M_n is increasing and

$$P(\sup M_n \le 1-\epsilon) = P(X_n \le 1-\epsilon \text{ for all } n) = 0.$$

Moreover, for $x > 0$,

$$P(a_n(1-M_n) \ge x) = P(M_n \le 1 - x/a_n) = \left(1 - \frac{x}{a_n}\right)^n.$$

So, take $a_n = n$ to get the exponential (1) limiting random variable. Uniform convergence holds because the limiting d. f. is continuous (see the proof of the Glivenko–Cantelli theorem.)