Math 235A, Fall 2025

Homework 7, Solution sketches

1. For $k \le n$,

$$P(T_n > k) = \frac{2^k\binom{n}{k}}{\binom{2n}{k}} = \prod_{i=1}^{k-1} \frac{2n-2i}{2n-i}.$$

Note that the above probability is 0 for $k > n$. Also, the expected number of pairs among the k socks chosen is $n\binom{k}{2}/\binom{2n}{2} = k(k-1)/(2(2n-1))$, so k should be on the order of $\sqrt{n}$. With this in mind, we write

$$\begin{aligned}
\log P(T_n > k) &= \sum_{i=1}^{k-1} \left[\log\left(1 - \frac{i}{n}\right) - \log\left(1 - \frac{i}{2n}\right)\right] \\
&= \sum_{i=1}^{k-1} \left[-\frac{i}{n} + \frac{i}{2n} + \mathcal{O}\left(\frac{k^2}{n^2}\right)\right] \\
&= -\frac{(k-1)(k-2)}{4n} + \mathcal{O}\left(\frac{k^3}{n^2}\right) \\
&= -\frac{k^2}{4n} + \mathcal{O}\left(\frac{k}{n} + \frac{k^3}{n^2}\right).
\end{aligned}$$

It follows that, for $x > 0$,

$$\log P(T \ge x\sqrt{n}) = -\frac{1}{4}x^2 + \mathcal{O}(n^{-1/2}),$$

and $T_n/\sqrt{n} \xrightarrow{d} X$, with $P(X > x) = e^{-x^2/4}$ for $x > 0$.

Moreover, for $k \le n$,

$$P(T_n > k) = \prod_{i=1}^{k-1}\left(1 - \frac{i}{2n-i}\right) \le \exp\left(-\sum_{i=1}^{k-1} \frac{i}{2n-i}\right) \le \exp\left(-\frac{k(k-1)}{4n}\right).$$

This bound holds (trivially) also when $k > n$. So, $P(T_n > x) \le P(T_n > \lfloor x \rfloor) \le e^{-x^2/(8n)}$ if $x > 10$, say. Finally,

$$E(T_n^2/n) = \int_0^\infty P(T_n > \sqrt{x}\sqrt{n})\,dx \le 1 + \int_1^\infty P(T_n > \sqrt{x}\sqrt{n}) \le 1 + \int_1^\infty e^{-x/8}\,dx,$$

for $n > 100$. Hence $T_n/\sqrt{n}$ are u.i. and $E(T_n)/\sqrt{n} \to \int_0^\infty e^{-x^2/4}\,dx = \sqrt{\pi}$.

2. Let $Y_n = |S_n/\sqrt{n}|^\alpha$. It suffices to show that $\sup_n E(Y_n^{4/\alpha}) < \infty$. But $Y_n^{4/\alpha} = S_n^4/n^2$ and

$$\begin{aligned}
E(S_n^4) &= \sum_{i=1}^n E(X_i^4) + \binom{4}{2}\sum_{i<j} E(X_i^2)E(X_j^2) \\
&= nEX_1^4 + 3n(n-1)(EX_1^2)^2 \le nEX_1^4 + 3n(n-1)(EX_1^4) < 3n^2 EX_1^4,
\end{aligned}$$

by Cauchy-Schwarz. So $E(Y_n^{4/\alpha}) \le 3EX_1^4$.