Math 235A, Fall 2025

Homework 8, Solution sketches

Durrett, 3.3.2.(i) By Fubini,

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt &= \frac{1}{2T}\int_{-T}^{T} dt\, e^{-ita}\int_{\mathbb{R}} e^{itx}\,d\mu(x)\\ &= \int_{\mathbb{R}} d\mu(x)\,\frac{1}{2T}\int_{-T}^{T} e^{it(x-a)}\,dt\\ &= \int_{\mathbb{R}} d\mu(x)\,\frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))\,dt.\end{aligned}$$

Let

$$I(x,T)=\frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))\,dt.$$

Then clearly $|I(x,T)|\le 1$ for all x and t. Moreover, by computing the integral we see that, as $T\to\infty$, $I(x,T)\to 1$ if $x=a$ and $I(x,T)\to 0$ otherwise. Now use DCT.

Durrett, 3.3.3. As $\varphi_{X-Y}=\varphi_X\varphi_{-Y}=\varphi_X\overline{\varphi_Y}=\varphi_X\overline{\varphi_X}=|\varphi_X|^2$, the first equality follows from the previous problem. To prove the second equality, let x_k be an enumeration of points x with $P(X=x)>0$. Then

$$\begin{aligned}P(X=Y) &= \int_{\mathbb{R}} d\mu(x)\int_{\mathbb{R}} 1_{\{x=y\}}\,d\mu(y)\\ &= \int_{\mathbb{R}} P(X=x)\,d\mu(x)\\ &= \int_{\mathbb{R}}\sum_k P(X=x_k)1_{\{x_k\}}(x)\,d\mu(x)\\ &= \sum_k P(X=x_k)\int_{\mathbb{R}} 1_{\{x_k\}}(x)\,d\mu(x)\\ &= \sum_k P(X=x_k)^2,\end{aligned}$$

where we used Fubini (or MCT) in the penultimate step.

Durrett, 3.4.4. Write

$$\sqrt{S_n}-\sqrt{n}=\frac{S_n-n}{\sqrt{n}}\cdot\frac{1}{\sqrt{S_n/n}+1},$$

and the second factor converges a.s. to 1/2 by SSLN. (You can also directly apply the delta method, by writing $\sqrt{S_n}-\sqrt{n}=\sqrt{n}(\sqrt{S_n/n}-1)$.)

Durrett, 3.4.5. Divide the top and the bottom by $\sigma\sqrt{n}$ and use SLLN.

1. (a) Omitted. (b) This is the probability that Poisson(n) random variable is at most its expectation n, so the limit is 1/2 by (a).

2. Immediately we get $E(X) = 0$. Let $\sigma^2 = EX^2 > 0$. By iterating the distributional equality we get, for any $n \geq 0$,

$$X \stackrel{d}{=} 2^{-n/2}(X_1 + \ldots + X_{2^n}),$$

where X_i are i.i.d. and $X_i \stackrel{d}{=} X$. Divide both sides by σ, then send $n \to \infty$ and use CLT to get that X/σ is standard Normal.