Math/Stat 235A, Fall 2025

Midterm Exam

Due: Wednesday, Nov. 5, 11:59pm, in Gradescope.

Directions: Work on these problems *alone*; you cannot discuss any part of the exam with anybody or use any books, papers, web sites, etc.; <u>you can consult only your notes from 235A</u>. Try to give concise solutions; none of the problems requires a long argument or computation. To facilitate grading, please solve each problem on a separate page, and select the corresponding problem number for each question when you upload to Gradescope. I will not reply to specific questions about exam problems. However, if you think that a problem is misstated, please let me know. You will receive extra credit if you are the first person to spot a mistake and I will post any corrections or clarifications on the course's web page.

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $A_i, B_i \in \mathcal{F}$ be such that $B_i \subset A_i$, $i = 1, 2, \ldots$ Prove the following generalization of subadditivity:

$$P(\cup_i A_i) - P(\cup_i B_i) \leq \sum_i (P(A_i) - P(B_i)).$$

2. Assume that X is a random variable such that $X \stackrel{d}{=} X^2$. Show that $P(X \in \{0, 1\}) = 1$. (*Hint*. To show that $P(X \leq 1) = 1$, for example, cover $(1, \infty)$ with countably many intervals of the form $(a, a^2]$, $a > 1$.)

3. Let T be the exponential random variable with expectation 1, that is, T has density $f_T(t) = e^{-t} 1_{[0,\infty)}$. Define the following two random variables,

$$\begin{aligned} N &= \lfloor T \rfloor = \sup\{n \in \mathbb{Z} : n \leq T\}, \\ X &= T - N, \end{aligned}$$

which are the integer and fractional part of T, respectively.

(a) Show that N and X are independent and determine their distributions. (*Hint*. Compute $P(N = k, X \leq x)$ for all relevant k and x.)

(b) Compute EN, EX and $E(XN)$. (*Hint*. Try to do it with minimal computations.)

(c) Compute

$$\lim_{\epsilon \to 0} \frac{P(0 < XN < \epsilon)}{\epsilon}.$$

(d) Assume $T_1, T_2, \ldots$ are i.i.d. distributed as T and $N_n = \lfloor T_n \rfloor$, $X_n = T_n - N_n$ for all $n \geq 1$. For which $\gamma > 0$ do we have $P(0 < X_n N_n < 1/n^\gamma \text{ i.o.}) = 1$?

4. Assume that $a < b$ and that X is a random variable with values in $[a, b]$. Prove that

$$\text{Var}(X) \leq \frac{(b-a)^2}{4}$$

and determine when equality holds. (*Hint*. Show that $\text{Var}(\alpha X + \beta) = \alpha^2 \text{Var}(X)$ and then use this to reduce the problem to $[a, b] = [-1, 1]$.)

5. All random variables in this problem are on a common probability space. For two random variables X and Y, we define

$$d(X,Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right).$$

(*Hint*. In the solutions, use properties of the function $\phi : x \mapsto x/(1+x)$ and its inverse on appropriate domains to avoid any lengthy arguments.)

(a) Show that this is a metric on the space of random variables: $d(X,Y) \in [0,\infty)$ and $d(X,Y) = 0$ if and only if $X = Y$ a.s.; $d(X,Y) = d(Y,X)$; and $d(X,Z) \le d(X,Y) + d(Y,Z)$ (known as the triangle inequality).

(b) Show that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$.

(c) Show that the there is no metric ρ on the space of random variables such that $X_n \to X$ a.s. if and only if $\rho(X_n, X) \to 0$. (*Hint*. Take a sequence that converges in probability. Use an argument by contradiction and subsequence extraction to show that existence of ρ would force the sequence to converge a.s.)

6. Let X_n be i.i.d. random variables with $P(X_n = 2^k) = 2^{-k}$, $k = 1, 2, \ldots$, and let $S_n = X_1 + \cdots + X_n$. Let $b_n = n \log_2 n$. We proved in class that $S_n/b_n \to 1$ in probability. Show that this convergence does not hold a.s. by proving the following claims.

(a) Prove that, for every $M > 0$, $\sum_n P(X_1 \ge M b_n) = \infty$.

(b) Using (a), prove that $\limsup X_n/b_n = \infty$ a.s.

(c) Using (b), conclude that $\limsup S_n/b_n = \infty$ a.s.

(d) Prove that $\liminf S_n/b_n \le 1$ a.s. (*Hint*: convergence in probability.)

7. Assume that $f, g : \mathbb{R} \to \mathbb{R}$ are nondecreasing bounded functions and that X is a random variable. Prove that

$$E(f(X)g(X)) \ge E(f(X)) \cdot E(g(X)).$$

(*Hint*. Verify that this holds when $f = 1_{[u,\infty)}$ and $g = 1_{[v,\infty)}$ for arbitrary u and v. Then reduce the problem to distribution functions f and g and use the Lebesgue-Stieltjes measures generated by them.)