Doob decomposition and martingales with bounded increments

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Doob decomposition

Theorem

Assume that X_n, $n \geq 0$ is a submartingale w.r.t. $\mathcal{F}_n$. Then X_n can be written in a unique way (up to a.s. equality) as $X_n = M_n + A_n$, where

1. *M_n is a martingale; and*
2. *A_n is increasing, predictable, and $A_0 = 0$.*

Note that $A_\infty = \lim_{n\to\infty} A_n$ exists (but it may be ∞).

Doob decomposition

Proof.

Assuming the decomposition:

$$\begin{aligned} E[X_n \mid \mathcal{F}_{n-1}] &= E[M_n \mid \mathcal{F}_{n-1}] + E[A_n \mid \mathcal{F}_{n-1}] \\ &= M_{n-1} + A_n \\ &= X_{n-1} + A_n - A_{n-1} \end{aligned}$$

and so

$$A_n = \sum_{k=1}^{n} \left(E[X_k \mid \mathcal{F}_{k-1}] - X_{k-1} \right),$$

and we have proved uniqueness. We need to check that so *defined* A_n are increasing, predictable, with $A_0 = 0$ (all clear), and that $X_n - A_n$ is a martingale:

$$E[X_n - A_n \mid \mathcal{F}_{n-1}] = E[X_n \mid \mathcal{F}_{n-1}] - A_n = X_{n-1} - A_{n-1}.$$

□

Doob decomposition

Now let X_n be a martingale with $X_0 = 0$, $EX_n^2 < \infty$ for all n. Then X_n^2 is a submartingale, so by Doob, $X_n^2 = M_n + A_n$, where M_n is a martingale and

$$\langle X \rangle_n := A_n = \sum_{k=1}^{n} \left(E[X_k^2 \mid \mathcal{F}_{k-1}] - X_{k-1}^2 \right)$$

is called the *bracket*, or *quadratic variation*, process for X_n.

It is used to measure fluctuations in X_n so it has a similar role to variance of a r.v.

Example. Let ξ_i be independent with $E\xi_i = 0$ and $E\xi_i^2 < \infty$, $S_n = \xi_1 + \cdots + \xi_n$. Then

$$E[S_n^2 \mid \mathcal{F}_{n-1}] = E[S_{n-1}^2 + 2S_{n-1}\xi_n + \xi_n^2 \mid \mathcal{F}_{n-1}] = S_{n-1}^2 + E\xi_n^2,$$

so $\langle S \rangle_n = \sum_{k=1}^{n} E\xi_k^2$ is deterministic in this case.

Doob decomposition

If N is a stopping time, then

$$X_{n\wedge N}^2 = M_{n\wedge N} + A_{n\wedge N}$$

and $M_{n\wedge N}$ is a martingale, while $A_{n\wedge N}$ is predictable, as

$$A_{n\wedge N} = \sum_{k=1}^{n-1} A_k 1_{\{N=k\}} + A_n 1_{\{N\geq n\}} = \sum_{k=1}^{n-1} A_k 1_{\{N=k\}} + A_n 1_{\{N\leq n-1\}^c}$$

It follows that the bracket of $X_{n\wedge N}$ is $\langle X\rangle_{n\wedge N}$.

Doob decomposition

Assume until further notice that X_n is a martingale with $X_0 = 0$, $EX_n^2 < \infty$ for all n, and $A_n = \langle X \rangle_n$.

Proposition

$E[\sup_{k\geq 1} X_k^2] \leq 4EA_\infty$.

Proof.

We first use the L^2-maximum inequality

$$E[\sup_{1\leq k\leq n} X_k^2] \leq 4EX_n^2 = 4EA_n \leq 4EA_\infty$$

and then MCT. □

Doob decomposition

Proposition

On $\{A_\infty < \infty\}$, $\lim X_n$ *exists and is finite a.s.*

Proof.

Fix $a > 0$ and let $N = \inf\{n : A_{n+1} > a\}$, which is a stopping time (by predictability of A_n). Then $X^2_{n\wedge N} = M_{n\wedge N} + A_{n\wedge N}$, and so by the previous proposition,

$$E[\sup_n X^2_{n\wedge N}] \leq 4EA_N \leq 4a.$$

Therefore, $\lim_{n\to\infty} X_{n\wedge N}$ exists a.s. and in L^2, and then $\lim_{n\to\infty} X_n$ exists a.s. on $\{N = \infty\} = \{A_\infty \leq a\}$, and finally $\lim_{n\to\infty} X_n$ exists a.s. on $\{A_\infty < \infty\} = \cup_{a=1}^{\infty}\{A_\infty \leq a\}$. □

Example. Let

$$\xi_n = \begin{cases} \pm 2^n & \text{w.p. } 1/2^{n+1} \\ 0 & \text{w.p. } 1 - 1/2^n \end{cases}$$

and assume these are independent and $S_n = \xi_1 + \ldots + \xi_n$. Then

$$\langle S \rangle_n = \sum_{k=1}^{n} 2^{2k} 2^{-k} = \sum_{k=1}^{n} 2^k \to \infty,$$

but S_n converges a.s. as $P(\xi_n \neq 0 \text{ i.o.}) = 0$.

So it is not in general true that X_n does not converge on $\{A_\infty = \infty\}$.

Martingales with bounded increments

Theorem

Assume that X_n is a martingale and that there exists a (deterministic) constant M such that $|X_{n+1} - X_n| \leq M$ for all $n \geq 0$. Let

$$C = \{\lim_n X_n \text{ exists and is finite}\}$$
$$D = \{\limsup_n X_n = \infty, \liminf_n X_n = -\infty\}$$

Then:

(a) $C = \{A_\infty < \infty\}$ *a.s.; and*

(b) $P(C \cup D) = 1$.

Martingales with bounded increments

Proof.

Assume WLOG that $X_0 = 0$.
To prove (b), define, for $K > 0$, the stopping time $N = N_K = \inf\{n : X_n \geq K\}$. Then $X_{n \wedge N}$ is a martingale and, by bounded increments, $X_{n \wedge N} \leq K + M$. Thus $\lim X_{n \wedge N}$ exists a.s. and is finite. So, $\lim X_n$ exists and is finite a.s. on

$$\cup_{K=1}^{\infty}\{N_K = \infty\} = \{\sup X_n < \infty\} = \{\limsup X_n < \infty\}.$$

By symmetry, $\lim X_n$ also exists and is finite a.s. on $\{\liminf X_n > -\infty\}$.

Martingales with bounded increments

Proof, continued.

To prove (a), it is now equivalent to show (by (b) and the previous proposition) that

$$P(A_\infty = \infty, \sup |X_n| < \infty) = 0.$$

Let now $N = N_K = \inf\{n : |X_n| \geq K\}$. We know that $E[X^2_{n\wedge N} - A_{n\wedge N}] = 0$ and $|X_{n\wedge N}| \leq K + M$.
So, $EA_{n\wedge N} \leq (K+M)^2$.
By MCT, $EA_N \leq (K+M)^2$.
So on $\{N = \infty\}$, $A_\infty < \infty$ a.s., that is, $P(N_K = \infty, A_\infty = \infty) = 0$ and then

$$P(\cup_{K=1}^{\infty}\{N_K = \infty\}, A_\infty = \infty) = 0,$$

but $\cup_{K=1}^{\infty}\{N_K = \infty\} = \{\sup |X_n| < \infty\}$. □

Martingales with bounded increments

Proposition (Borel-Cantelli part 3)

Let $\mathcal{F}_n$ be a filtration and $A_n \in \mathcal{F}_n$, $n \geq 1$. Then, a.s.,

$$\{A_n \ i.o.\} = \left\{ \sum_{n=1}^{\infty} E[1_{A_n} \mid \mathcal{F}_{n-1}] = \infty \right\}$$

Observe that the first two Borel-Cantelli lemmas follow from this one.

Martingales with bounded increments

Proof.

Define

$$X_n = \sum_{k=1}^{n} \left(1_{A_k} - E[1_{A_k} \mid \mathcal{F}_{k-1}]\right)$$

(with $X_0 = 0$). This is a martingale as

$$E[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] = E[1_{A_n} - E[1_{A_n} \mid \mathcal{F}_{n-1}] \mid \mathcal{F}_{n-1}] = 0.$$

Clearly $|X_n - X_{n-1}| \leq 1$. We apply the previous theorem, noting that on both C and D

$$\sum_{k=1}^{\infty} 1_{A_k} = \infty \iff \sum_{k=1}^{\infty} E[1_{A_k} \mid \mathcal{F}_{k-1}] = \infty.$$

□

Three series theorem

Theorem (Three series theorem)

Let $\xi_1, \xi_2, \ldots$ be independent r.v.'s. For $A > 0$, let $\eta_n = \xi_n 1_{\{|\xi_n| \le A\}}$. Consider the following three series (i)$\sum_{n=1}^{\infty} P(|\xi_n| \ge A)$; (ii)$\sum_{n=1}^{\infty} E\eta_n$; and (iii)$\sum_{n=1}^{\infty} \mathrm{Var}(\eta_n)$. Then the following are equivalent:

(a) $\sum_{n=1}^{\infty} \xi_n$ *converges a.s.;*

(b) *(i), (ii), and (iii) converge for all $A > 0$; and*

(c) *(i), (ii), and (iii) converge for some $A > 0$.*

Recall that $\{\sum_{n=1}^{\infty} \xi_n$ converges$\}$ is a tail event, so by the 0-1 law, the statement (a) is equivalent to $P(\sum_{n=1}^{\infty} \xi_n\}$ converges$) > 0$.

Three series theorem

Example. Assume that ϵ_n are i.i.d., with $P(\epsilon_n = \pm 1) = \frac{1}{2}$. When does $S = \sum_{n=1}^{\infty} \epsilon_n \frac{1}{n^p}$ converge?

We only need to check when (iii) converges (with ξ_i instead of η_i). This gives the condition

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} < \infty \iff p > 1/2$$

If $0 < p \leq 1/2$, the partial sums $S_n = \sum_{k=1}^{n} \epsilon_k \frac{1}{k^p}$ are a.s. dense in $\mathbb{R}$. This follows because S_n is a martingale with bounded increments.

When $p > 1/2$, we can determine the distribution of S by its characteristic function

$$E[e^{itS}] = E[e^{it\sum_n \epsilon_n n^{-p}}] = \prod_{n=1}^{\infty} E[e^{it\epsilon_n n^{-p}}] = \prod_{n=1}^{\infty} \cos(tn^{-p}).$$

Three series theorem

Proof of Three series theorem.

(c)$\Longrightarrow$ (a)
$S_n = \sum_{k=1}^{n}(\eta_k - E\eta_k)$ is a martingale with $\langle S\rangle_n = \sum_{k=1}^{n} \mathrm{Var}(\eta_k)$.
By convergence of (iii), $\langle S\rangle_\infty < \infty$, and so S_n converges a.s.
Then, as (ii) converges, $\sum_{k=1}^{\infty} \eta_k$ converges a.s.
By convergence of (i), $P(\xi_k \neq \eta_k \text{ i.o.}) = 0$, and so $\sum_{k=1}^{\infty} \xi_k$ converges a.s.

(b)$\Longrightarrow$ (c)
Trivial.

Three series theorem

Proof of Three series theorem, continued.

(a)$\Longrightarrow$ (b)
Fix an $A > 0$. If (i) diverges, then $P(|\xi_k| > A \text{ i.o.}) = 1$ and so $\sum \xi_k$ diverges a.s. So, (i) must converge and then $P(\xi_k \neq \eta_k \text{ i.o.}) = 0$ so that $\sum \eta_k$ converges a.s.

Now we apply the "symmetrization trick." Let $\eta_k \stackrel{d}{=} \eta_k'$, and construct the sequence (η_k') of independent r.v.'s, which is independent of (η_k).

Then $\sum_k \eta_k'$ also converges a.s. and then so does $\sum_k (\eta_k - \eta_k')$. Now $S_n' = \sum_{k=1}^n (\eta_k - \eta_k')$ is a martingale with bounded increments, and so its bracket $\langle S' \rangle_n = \sum_{k=1}^n \mathrm{Var}(\eta_k - \eta_k') = \sum_{k=1}^n 2\mathrm{Var}(\eta_k)$ must converge. It follows that $\sum_{k=1}^\infty \mathrm{Var}(\eta_k) < \infty$. So, $\sum_{k=1}^n (\eta_k - E\eta_k)$ converges a.s., as its bracket converges. Thus, $\sum_{k=1}^\infty E\eta_k$ converges.

□