Uniform integrability and martingales

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Uniform integrability

Recall that a family $\{X_n : n\}$ of random variables is *uniformly integrable* if

$$\lim_{M\to\infty} \sup_n E[|X_n| 1_{\{|X_n| \geq M\}}] = 0$$

Here, the index set is arbitrary (but we assume natural numbers in convergence statements). The random variables do not need to be defined on the same probability space.

Uniform integrability

Theorem

(0) *If* $|X_n| \le Y_n$, *and* Y_n *are u.i., then* X_n *are u.i.*

(1) *If* X_n *are u.i., then* $\sup_n E|X_n| < \infty$.

(2) *If* $\sup_n E|X_n|^\alpha < \infty$ *for some* $\alpha > 1$, *then* X_n *are u.i.*

(3) *If* $X_n \overset{d}{=} X$ *and* $E|X| < \infty$, *then* X_n *are u.i.*

(4) *If* X_n *and* Y_n *are u.i., so is* $X_n + Y_n$.

(5) *If* X_n *are u.i.,* $X_n \xrightarrow{d} X$, *then* $EX_n \to EX$.

(6) *If* $E|X_n| \to 0$, $E|X_n| < \infty$ *for all n, then* X_n *are u.i.*

(7) *If* $X_n \to X$ *a.s.,* $E|X_n| \to E|X|$, $E|X_n| < \infty$ *for all n, and* $E|X| < \infty$, *then* X_n *are u.i.*

Uniform integrability

Proof.

(0)–(5) are routine, except perhaps (4):

$$|X_n + Y_n| 1_{\{|X_n+Y_n| \geq M\}} \leq 2|X_n| 1_{\{|X_n| \geq M/2\}} + 2|Y_n| 1_{\{|Y_n| \geq M/2\}}$$

For example, (5) is proved by assuming a.s. convergence. By (3) and (4), $Y_n = |X_n - X|$ are u.i. For every M, $E(Y_n 1_{\{Y_n \leq M\}}) \to 0$ by DCT, and then
$\limsup EY_n \leq \sup_n E(Y_n 1_{\{Y_n \geq M\}}) \to 0$ as $M \to \infty$.
So, $\limsup EY_n = 0$.

To prove (6), fix $\epsilon > 0$. Choose $N = N_\epsilon$ so that $E|X_n| < \epsilon$ for $n \geq N_\epsilon$. Then choose $M = M_\epsilon$ so that $E[|X_n| 1_{\{|X_n| \geq M\}}] \leq \epsilon$ for $n = 1, \ldots, N$.

Proof, continued.

To prove (7), pick $\epsilon > 0$. Let $\varphi = \varphi_M$ be the function on $[0, M]$ that linearly connects $(0, 0)$ and $(M - 1, M - 1)$, and $(M - 1, M - 1)$ and $(M, 0)$. Choose M so that $E\varphi(|X|) \geq E|X| - \epsilon/2$ (MCT). As $E\varphi(|X_n|) \to E\varphi(|X|)$ (DCT), $E\varphi(|X_n|) \geq E|X| - \epsilon$ for $n \geq N_\epsilon$. Then for $n \geq N_\epsilon$,

$$
\begin{aligned}
E[|X_n| 1_{\{|X_n| \geq M\}}] &= E|X_n| - E[|X_n| 1_{\{|X_n| < M\}}] \\
&\leq E|X_n| - E\varphi(|X_n|) \\
&\leq E|X_n| - E|X| + \epsilon < 2\epsilon,
\end{aligned}
$$

for $n \geq N_\epsilon$ (possibly enlarged). The remaining r.v.'s $X_1, \ldots X_{N_\epsilon - 1}$ are dealt with as before. □

Uniform integrability of conditional expectation

Theorem

Assume that X is a r.v. on $(\Omega, \mathcal{F}, P)$ with $E|X| < \infty$. Then

$$\{E[X \mid \mathcal{G}] : \mathcal{G} \subset \mathcal{F} \text{ is a } \sigma\text{-algebra}\}$$

is a u.i. family.

Proof.

Step 1. If $P(A_n) \to 0$, then $E(|X| 1_{A_n}) \to 0$.
If not, there exists a sequence A_n with $E(|X| 1_{A_n}) \geq \delta > 0$ and $P(A_n) \leq 1/2^n$. But then by BC, $1_{A_n} \to 0$ a.s., then $|X| 1_{A_n} \to 0$ a.s., and then $E[|X| 1_{A_n}] \to 0$ by DCT, contradiction.

Step 2. $P(E[|X| \mid \mathcal{G}] \geq M) \leq \frac{1}{M} E|X|$.
This is Markov's inequality.

Uniform integrability of conditional expectation

Proof, continued.

Step 3.

$$\begin{aligned} & E[|E[X \mid \mathcal{G}]|1_{\{|E[X|\mathcal{G}]| \geq M\}}] \\ & \leq E[E[|X| \mid \mathcal{G}]1_{\{E[|X||\mathcal{G}] \geq M\}}] \\ & = E[|X|1_{\{E[|X||\mathcal{G}] \geq M\}}] \end{aligned}$$

Pick $\epsilon > 0$. Choose a $\delta > 0$ so that $P(A) < \delta$ implies $E(|X|1_A] < \epsilon$ (Step 1). Then choose M so that $E|X|/M < \delta$ and use Step 2. □

Uniform integrability of conditional expectation

Corollary

If $\xi_1, \xi_2, \ldots$ *are i.i.d. with finite expectation* μ, *and* $S_n = \xi_1 + \cdots + \xi_n$, *then*

$$E\left|\frac{S_n}{n} - \mu\right| \to 0.$$

Proof.

As $S_n/n = E[\xi_1|S_n]$, S_n/n are u.i. □

Uniform integrability of submartingales

Theorem

Assume X_n is a submartingale. TFAE:

(i) X_n *u.i.*

(ii) X_n *converge a.s. and in* L^1*. (There exist* $X \in L^1$ *so that* $X_n \to X$ *a.s. and* $E|X_n - X| \to 0$*.)*

(iii) X_n *converge in* L^1*. (There exist* $X \in L^1$ *so that* $E|X_n - X| \to 0$*.)*

Proof.

((i)$\Longrightarrow$(ii)) (i) also implies a.s. convergence by martingale CT and (1) of the previous theorem. Now (5) of the same theorem implies (ii).
((iii)$\Longrightarrow$(i)) (iii) also implies a.s. convergence by martingale CT, and (7) of the previous theorem implies (i). $\square$

Uniform integrability of martingales

Theorem

Assume X_n is a martingale. TFAE:

(i) X_n *u.i.*

(ii) X_n *converge a.s. and in* L^1.

(iii) X_n *converge in* L^1.

(iv) *There exists an* $X \in L^1$ *so that* $X_n = E[X \mid \mathcal{F}_n]$.

Proof.

((ii)$\Longrightarrow$(iv)) Let $X = \lim X_n$. Then $E[X_m \mid \mathcal{F}_n] = X_n$ for $m \geq n$ means that, for all $A \in \mathcal{F}_n$ and $m \geq n$, $E[X_m 1_A] = E[X_n 1_A]$. But

$$|E[X_m 1_A] - E[X 1_A]| \leq E|X_m - X| \to 0$$

as $m \to \infty$, and so $E[X 1_A] = E[X_n 1_A]$. □

Uniform integrability of martingales

Theorem

Assume that $E|X| < \infty$ *and* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ *is a filtration. Let* $\mathcal{F}_\infty = \sigma(\cup_{n \geq 1} \mathcal{F}_n)$. *Then*

$$E[X \mid \mathcal{F}_n] \to E[X \mid \mathcal{F}_\infty]$$

a.s. and in L^1.

Proof.

The sequence $X_n = E[X \mid \mathcal{F}_n]$ is a u.i. martingale, hence $X_n \to Y = \limsup X_n$ a.s. and in L^1. Note that Y is $\mathcal{F}_\infty$-measurable. By previous proof, $E[X \mid \mathcal{F}_n] = E[Y \mid \mathcal{F}_n]$ for all n, and so $E[X 1_A] = E[Y 1_A]$ for all $A \in \cup_{n \geq 1} \mathcal{F}_n$, which is a π-system that generates $\mathcal{F}_\infty$. □

Uniform integrability of martingales

Corollary (Lévy 0-1 law)

Assume $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ is a filtration. For any $A \in \mathcal{F}_\infty = \sigma(\cup_{n\geq 1}\mathcal{F}_n)$,

$$E[1_A \mid \mathcal{F}_n] \to 1_A \text{ a.s.}$$

Observe that this is more general than Kolmogorov 0-1 law: in that context, if A is in the tail σ-algebra, it is independent of each $\mathcal{F}_n$ and so $E[1_A \mid \mathcal{F}_n] = P(A)$.

Uniform integrability of martingales

Theorem

Assume that $E|X| < \infty$ *and* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ *is a filtration. Let* $\mathcal{F}_\infty = \sigma(\cup_{n\geq 1}\mathcal{F}_n)$*. Assume that a sequence of r.v.'s* X_n *converges to* X *a.s. and* $Y = \sup_n |X_n| \in L^1$*. Then*

$$E[X_n \mid \mathcal{F}_n] \to E[X \mid \mathcal{F}_\infty]$$

a.s. and in L^1*.*

Uniform integrability of martingales

Proof.

Let $W_N = \sup\{|X_m - X_n| : m, n \geq N\}$. Note that $W_N \leq 2Y$. So, by the previous theorem, for all N,

$$\limsup_n E[|X_n - X| \mid \mathcal{F}_n] \leq \limsup_n E[W_N \mid \mathcal{F}_n] = E[W_N \mid \mathcal{F}_\infty].$$

As $N \to \infty$, $W_N \downarrow 0$ a.s. By conditional DCT, $E[W_N \mid \mathcal{F}_\infty] \downarrow 0$. Then

$$\limsup_n |E[X_n \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_n]| \leq \limsup_n E[|X_n - X| \mid \mathcal{F}_n] = 0$$

a.s. The L^1 convergence follows from

$$\begin{aligned} &E|E[X_n \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_\infty]| \\ &\leq E|E[X_n \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_n]| + E|E[X \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_\infty]| \\ &\leq E|X_n - X| + E|E[X \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_\infty]| \to 0. \end{aligned}$$

□