Martingales and optional stopping

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MAT 235B

February 12, 2024

Optional stopping

So far, we know that if X_n is a submartingale, and N is an arbitrary stopping time, then

$$EX_0 \leq EX_{N\wedge n} \leq X_n.$$

Lemma

Assume that X_n is a submartingale, and N is a stopping time. If $E(|X_N|1_{\{N<\infty\}}) < \infty$ and $|X_n|1_{\{N>n\}}$ are u.i., then $X_{N\wedge n}$ is u.i.

Proof.

$$|X_{N\wedge n}| = |X_N|1_{\{N\leq n\}} + |X_n|1_{\{N>n\}} \leq |X_N|1_{\{N<\infty\}} + |X_n|1_{\{N>n\}}$$

□

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Lemma

Assume that X_n is a u.i. submartingale, and N is a stopping time. Then $X_{N\wedge n}$ is also u.i.

Proof.

As X_n^+ is a submartingale, $EX_{N\wedge n}^+ \leq EX_n^+ \leq E|X_n|$. Also, $\sup_n E|X_n| < \infty$, so by martingale CT, $X_{N\wedge n}$ converges a.s. as $n \to \infty$ to X_N and $E|X_N| < \infty$. Then use $|X_N|1_{\{N<\infty\}} \leq |X_N|$ and $|X_n|1_{\{N>n\}} \leq |X_n|$. □

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Lemma

Assume that X_n is a u.i. submartingale, with $X_\infty = \lim X_n$, and $N \le \infty$ is a stopping time. Then $EX_0 \le EX_N \le EX_\infty$.

Proof.

$$EX_0 \le EX_{N \wedge n} \le EX_n.$$

and $EX_{N \wedge n} \to EX_N$, and $EX_n \to EX_\infty$, both by u.i. □

Theorem

Assume X_n is a submartingale and $L \le M$ are stopping times. If $X_{M \wedge n}$ is u.i., then $EX_L \le EX_M$.

Proof.

$Y_n = X_{M \wedge n}$ is a u.i. submartingale and L is a stopping time. So $EY_0 \le EY_L \le EY_\infty$. □

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Corollary

Assume X_n is a submartingale such that $E[|X_{n+1} - X_n| \mid \mathcal{F}_n] \le B \in \mathbb{R}$, and N is a stopping time with $EN < \infty$. Then $X_{N \wedge n}$ is u.i. and so $EX_N \ge EX_0$.

Proof.

$$X_{N\wedge n} = |X_0 + \sum_{k=1}^{n}(X_k - X_{k-1})1_{\{N\ge k\}}|$$
$$\le |X_0| + \sum_{k=1}^{\infty}|X_k - X_{k-1}|1_{\{N\ge k\}}$$

We claim that RHS is in L^1.

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Proof, continued.

$$\sum_{k=1}^{\infty} E(|X_k - X_{k-1}|1_{\{N \geq k\}})$$
$$= \sum_{k=1}^{\infty} E(E[|X_k - X_{k-1}|1_{\{N \geq k\}} \mid \mathcal{F}_{k-1}])$$
$$= \sum_{k=1}^{\infty} E(1_{\{N \geq k\}} E[|X_k - X_{k-1}| \mid \mathcal{F}_{k-1}])$$
$$\leq \sum_{k=1}^{\infty} B \cdot P(N \geq k) = B \cdot EN.$$

□

Example.
Assume that $\xi_1, \xi_2, \ldots$ are i.i.d., $P(\xi_i = 1) = p \in (1/2, 1)$, $P(\xi_i = -1) = 1 - p$. Then $S_n = \xi_1 + \cdots + \xi_n$ is a simple (asymmetric) random walk. Let

$$N = \inf\{n : S_n = -a \text{ or } S_n = b\}$$

We have

$$P(N > n) \leq \left(1 - p^{a+b}\right)^{\lfloor n/(a+b) \rfloor}$$

and so $EN < \infty$.

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Define $M_n = S_n - n(2p-1)$. This is a martingale, and $M_{n\wedge N}$ is u.i. by the previous corollary or directly by

$$M_{n\wedge N} \le \max\{a,b\} + N(2p-1).$$

So $EM_N = 0$ and

$$-aP(S_N = -a) + bP(S_n = b) - (2p-1)EN = 0.$$

Two unknowns; we need another martingale.

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Let

$$M_n = \left(\frac{1-p}{p}\right)^{S_n}$$

Note $\frac{1-p}{p} < 1$. Then M_n is a martingale, as

$$E\left(\frac{1-p}{p}\right)^{\xi_1} = \frac{1-p}{p} \cdot p + \frac{p}{1-p} \cdot (1-p) = 1.$$

Also, $M_{n \wedge N}$ is a bounded martingale, thus u.i.:

$$M_{n \wedge N} \le \left(\frac{1-p}{p}\right)^{-a}.$$

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It follows that

$$1 = EM_N = \left(\frac{1-p}{p}\right)^{-a} P(S_n = -a) + \left(\frac{1-p}{p}\right)^{b} P(S_n = b)$$

and so we get the *gambler's ruin* probability

$$P(S_N = b) = \frac{\left(\frac{1-p}{p}\right)^{-a} - 1}{\left(\frac{1-p}{p}\right)^{-a} - \left(\frac{1-p}{p}\right)^{b}}$$

$$P(S_N = -a) = \frac{1 - \left(\frac{1-p}{p}\right)^{b}}{\left(\frac{1-p}{p}\right)^{-a} - \left(\frac{1-p}{p}\right)^{b}}$$

$$\xrightarrow{b \to \infty} \left(\frac{1-p}{p}\right)^{a} = P(S_n \text{ ever reaches } -a).$$

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By sending $a \to \infty$, we see that $P(S_n \text{ ever reaches } b) = 1$.

From the first martingale, we can also get

$$EN = \frac{1}{2p-1}\left[-a \cdot \frac{1 - \left(\frac{1-p}{p}\right)^b}{\left(\frac{1-p}{p}\right)^{-a} - \left(\frac{1-p}{p}\right)^b} + b \cdot \frac{\left(\frac{1-p}{p}\right)^{-a} - 1}{\left(\frac{1-p}{p}\right)^{-a} - \left(\frac{1-p}{p}\right)^b}\right]$$

Now let $T = \inf\{n : S_n = 1\}$. We know that $P(T < \infty) = 1$. By taking $b = 1$ and $a \to \infty$ in EN, we get

$$ET = \frac{1}{2p-1}.$$

Optional stopping

Can we compute the distribution of T? For example,

$$P(T = 1) = p, P(T = 2) = 0, P(T = 3) = (1 - p)p^2, \ldots$$

Assume $\theta > 0$. Then

$$M_n = \frac{e^{\theta S_n}}{(Ee^{\theta \xi_1})^n}$$

is a martingale, as a product of mean-1 r.v.'s. By Jensen,

$$Ee^{\theta \xi_1} \geq e^{\theta E \xi_1} = e^{\theta(2p-1)} \geq 1$$

so $M_{n \wedge T} \leq e^{\theta}$, so it is u.i. and $EM_T = 1$, that is,

$$e^{-\theta} = E\left[(pe^{\theta} + (1 - p)e^{-\theta})^{-T}\right]$$

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Determine $s \in (0,1)$ so that

$$\frac{1}{s} = pe^{\theta} + (1-p)e^{-\theta},$$

and taking the root which is bigger than 1,

$$e^{\theta} = \frac{1+\sqrt{1-4p(1-p)s^2}}{2ps}.$$

$$\begin{aligned} E(s^T) &= \frac{2ps}{1+\sqrt{1-4p(1-p)s^2}} = \frac{1-\sqrt{1-4p(1-p)s^2}}{2(1-p)s} \\ &= \frac{1}{2(1-p)} \sum_{k=1}^{\infty} (-1)^{k-1} (4p(1-p))^k \binom{1/2}{k} s^{2k-1} \end{aligned}$$

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$$\binom{1/2}{k} = (-1)^{k-1}\binom{2k}{k}2^{-2k}\frac{1}{2k-1}$$

So, for $k \geq 1$,

$$\begin{aligned} P(T = 2k-1) &= (-1)^{k-1}(4p(1-p))^k\binom{1/2}{k} \\ &= \frac{(4p(1-p))^k}{2(1-p)}\binom{2k}{k}2^{-2k}\frac{1}{2k-1} \end{aligned}$$

This formula is true for all $p \in (0,1)$ as both sides are polynomials in p (of degree $2k-1$). Therefore, the formula for Es^T is also valid for all $p \in (0,1)$, provided we define, for all p, $Es^T = E(s^T 1_{\{T<\infty\}})$.
To check, by MCT,

$$P(T < \infty) = \lim_{s\uparrow 1} E(s^T) = \frac{1-|2p-1|}{2(1-p)s} = \begin{cases} 1 & p \geq 1/2 \\ \frac{p}{1-p} & p < 1/2 \end{cases}$$