Markov chains: recurrence and transience

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Recurrent and transient states

Assume X_n is a MC on a state space S. Fix $y \in S$.
Define

$$T_y^0 = 0$$
$$T_y^k = inf\{n > T_y^{k-1} : X_n = y\}$$

to be the times of successive visits to k (except possibly for $k = 0$). Let $T_y = T_y^1$ be the time of first visit to y after time 0 and

$$\rho_{xy} = P_x(T_y < \infty).$$

A state $a \in S$ is *absorbing* if $p(a, a) = 1$. For such a state $\rho_{ay} = 1_{a=y}$.

Example. If $S = \{a, b, c\}$, a and c are absorbing, and $p(b, \cdot) \equiv \frac{1}{3}$, then $\rho_{ac} = 0$, $\rho_{bc} = \frac{1}{2}$, $\rho_{cc} = 1$.

Recurrent and transient states

Proposition

For every $x, y, z \in S$*,* $k \geq 1$*,*

$$P_x(T_y^k < \infty) = \rho_{xy}\rho_{yy}^{k-1},$$
$$\rho_{xy} \geq \rho_{xz}\rho_{zy}$$

Proof.

Use SMP: go from x to y, then return to y $k-1$ times; go from x to z, then from z to y. □

A state $y \in S$ is *recurrent* if $\rho_{yy} = 1$ and *transient* if $\rho_{yy} < 1$.

Example. In the previous example, a and c are recurrent, while b is transient as $\rho_{bb} = \frac{1}{3}$.

Recurrent and transient states

Let $N(y) = \sum_{n=1}^{\infty} 1_{\{X_n=y\}}$ be the total no. of visits to y. By the proposition, if y is recurrent, then $P_y(T_y^k < \infty) = 1$ for all k, so that $P_y(X_n = y \text{ i.o.}) = P_y(N(y) = \infty) = 1$. We have

$$E_x N(y) = \sum_{n=1}^{\infty} p^n(x, y),$$

and on the other hand

$$\begin{aligned} E_x N(y) &= \sum_{k=1}^{\infty} P_x(N(y) \geq k) \\ &= \sum_{k=1}^{\infty} P_x(T_y^k < \infty) \\ &= \sum_{k=1}^{\infty} \rho_{xy} \rho_{yy}^{k-1} = \frac{\rho_{xy}}{1 - \rho_{yy}} \end{aligned}$$

Next proposition follows.

Recurrent and transient states

Proposition

The state y is recurrent iff

$$E_y N(y) = \sum_{n=1}^{\infty} p^n(y, y) = \infty.$$

Proposition (Recurrence is contagious)

Assume $x, y \in S$, $x \neq y$. If x is recurrent and $\rho_{xy} > 0$, then y is also recurrent and $\rho_{yx} = 1$, and so also $\rho_{xy} = 1$.

Recurrent and transient states

Proof.

Assume $\rho_{yx} < 1$. As $\rho_{xy} > 0$, there exists a $K \geq 1$ so that $p^K(x,y) > 0$; assume that K is the smallest such. As

$$p^K(x,y) = \sum_{y_1,\dots,y_{K-1} \in S} p(x,y_1) \cdots p(y_{K-1},y),$$

there exits $y_1, \dots, y_{K-1} \in S \setminus \{x,y\}$ so that $p(x,y_1) > 0, \dots p(y_{K-1},y) > 0$. Then

$$1 - \rho_{xx} = P_x(T_x = \infty) \geq p(x,y_1) \cdots p(y_{K-1},y)(1 - \rho_{yx}) > 0,$$

and x is not recurrent. This shows that $\rho_{yx} = 1$.

Proof, continued.

To show that y is recurrent, find L so that $p^L(y,x) > 0$. Then, for all $n \geq 1$,

$$p^{L+n+K}(y,y) \geq p^L(y,x)p^n(x,x)p^K(x,y)$$

and

$$\sum_{n=1}^{\infty} p^n(y,y) \geq \sum_{n=1}^{\infty} p^{L+n+K}(y,y) = \infty.$$

□

Recurrent and transient states

We say that a subset $C \subset S$ of states is *closed* if $x \in C$ and $\rho_{xy} > 0$ imply that $y \in C$. That is, for every $x \in C$, $P_x(X_n \in C \text{ for all } n) = 1$.

We say that a subset $D \subset S$ of states is *irreducible* if $\rho_{xy} > 0$ for all $x, y \in D$.

The previous proposition implies the following.

Corollary

The set of all recurrent sites is closed. If $D \subset S$ is irreducible, and contains at least one recurrent state, then all states in D are recurrent.

Recurrent and transient states

Proposition

If C is a finite closed set, then C includes at least one recurrent state.

Proof.

Take an $x \in C$. Then

$$\begin{aligned}\sum_{y \in C} E_x N(y) &= \sum_{y \in C} \sum_{n=1}^{\infty} p^n(x, y) \\ &= \sum_{n=1}^{\infty} \sum_{y \in C} p^n(x, y) \\ &= \sum_{n=1}^{\infty} P_x(X_n \in C) = \infty,\end{aligned}$$

so $E_x N(y) = \infty$ for at least one $y \in C$. □

Recurrent and transient states

A chain is *irreducible* if the entire state space S is irreducible. Irreducible chain is *recurrent* if one (equivalently, every) state is recurrent and *transient* otherwise.

Example. Let S_n be SRW on $\mathbb{Z}$ with probability $p \in (0, 1)$ of rightward jump. We know (by martingale arguments) that this chain is recurrent if $p = 1/2$, and that the chain is transient if $p \neq 1/2$. In fact, if $p > 1/2$,

$$\rho_{00} = 1 - p + p\, P_0(T_{-1} < \infty) = 1 - p + p\,\frac{1-p}{p} = 2(1-p).$$

Recurrent and transient states

Theorem (Decomposition theorem)

Let $R = \{x : \rho_{xx} = 1\}$ be the set of recurrent states. Then $R = \cup_i R_i$, a disjoint union of closed irreducible sets R_i.

Proof.

For $x, y \in R$ let $x \sim y$ iff $x = y$ or $\rho_{xy} > 0$. This is an equivalence relation on R. Any equivalence class is clearly closed and irreducible. □

Examples

Example. Branching process with offspring distribution ξ with $P(\xi = 0) > 0$. Then 0 is absorbing. Also, $\rho_{k0} > 0$ for all k. This alone implies that all states except 0 are transient. (But we already knew this.)

Example. M/G/1 queue. Recall that a_k is the probability that k customers arrive during service time. Let $\mu = \sum_{k=0}^{\infty} k a_k$ be the expected number of such customers. We assume that the server time is not deterministically zero, so that all $a_k > 0$, which implies that the chain is irreducible.

Think of arriving customers as children of the customer who is next in line when the serving starts (if there is such a customer). Assume $x \geq 1$. If $\mu > 1$, then $P_x(T_0 < \infty) = \rho^x$, where ρ is the probability that the branching process with offspring distribution given by a_k dies out. The chain is transient. If $\mu \leq 1$, then $P_x(T_0 < \infty) = 1$, the chain is recurrent.

Examples

Example. Let $S_n^{(d)}$ be the d-dimensional symmetric simple random walk, $S_n = S_n^{(1)}$.

We have, by Stirling

$$P_0(S_{2n} = 0) = \binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}},$$

which implies, one more time, recurrence of 1d SSRW.

Let B_n be a Binomial(n, p) r.v. Then

$$p_n = P(B_n \text{ is even}) = \frac{1}{2} + \frac{1}{2}(1 - 2p)^n,$$

because

$$p_{n+1} = (1 - p)p_n + p(1 - p_n) = p_n(1 - 2p) + p, \quad p_0 = 1.$$

Lemma

Let $\xi_1, \xi_2, \ldots$ be i.i.d. bounded with $E\xi_1 = \mu$, $S_n = \xi_1 + \ldots + \xi_n$. Then for every $\epsilon > 0$ there exists a $\gamma = \gamma_\epsilon > 0$, so that

$$P(S_n \notin [n(\mu - \epsilon), n(\mu + \epsilon)] \leq e^{-\gamma n}$$

Proof.

For $\lambda \in \mathbb{R}$, let $h(\lambda) = \lambda(\mu + \epsilon) - \log E[e^{\lambda \xi_1}]$. As ξ_1 is bounded, $h(\lambda)$ exists and is smooth for all λ, with
$h'(\lambda) = (\mu + \epsilon) - E[\xi_1 e^{\lambda \xi_1}]/E[e^{\lambda \xi_1}]$.
Observe that $h(0) = 0$ and $h'(0) = \epsilon > 0$, so there exists a $\lambda > 0$ so that $h(\lambda) > 0$. But by Markov,

$$P(S_n \geq n(\mu + \epsilon)) \leq P(e^{\lambda S_n - \lambda n(\mu+\epsilon)} \geq 1) \leq e^{-nh(\lambda)}.$$

The other inequality follows by replacing all ξ_i be $-\xi_i$. □

Examples

Assume $d = 2$. If the walk makes $2n$ steps, let B_{2n} be the number of steps in horizontal direction, which is Binomial$(2n, 1/2)$. Then for any $\epsilon > 0$ and a large enough n,

$$
\begin{aligned}
&P(S_{2n}^{(2)} = 0) \\
&= \sum_{k=0}^{n} P(B_{2n} = 2k) P(S_{2k} = 0) P(S_{2(n-k)} = 0) \\
&\geq \sum_{(1-\epsilon)n/2 \leq k \leq (1+\epsilon)n/2} P(B_{2n} = 2k) \left(\frac{1-\epsilon}{\sqrt{\pi(1+\epsilon)\frac{n}{2}}} \right)^2 \\
&\geq \frac{(1-\epsilon)^2}{1+\epsilon} \frac{2}{\pi n} \sum_{k=0}^{n} P(B_{2n} = 2k) - e^{-\gamma n} \sim \frac{(1-\epsilon)^2}{1+\epsilon} \frac{1}{\pi n}
\end{aligned}
$$

By the analogous upper bound, $P(S_{2n}^{(2)} = 0) \sim \frac{1}{\pi n}$, and the RW is recurrent. (In fact, $P(S_{2n}^{(2)} = 0) = P(S_{2n} = 0)^2$.)

Examples

Assume $d = 3$. Now the number of steps in the z-direction B_{2n} is Binomial $(2n, 1/3)$ and

$$\begin{aligned} & P(S_{2n}^{(3)} = 0) \\ & = \sum_{k=0}^{n} P(B_{2n} = 2k) P(S_{2k} = 0) P(S_{2(n-k)}^{(2)} = 0) \\ & \sim \sum_{k=0}^{n} P(B_{2n} = 2k) \frac{1}{\sqrt{\pi \frac{n}{3}}} \frac{1}{\pi \frac{2n}{3}} \\ & \sim \frac{3^{3/2}}{4\pi^{3/2}} \cdot \frac{1}{n^{3/2}} \end{aligned}$$

Therefore, the RW is transient and $\rho_{00} < 1$. In fact $\rho_{00} \approx 0.3405$, as can be computed by

$$\frac{\rho_{00}}{1 - \rho_{00}} = \sum_{n=1}^{\infty} P(S_{2n}^{(3)} = 0),$$

but there are better methods.

Example. Birth-death chain X_n, with $S = \{0, 1, 2 \ldots\}$, $p_i = p(i, i+1) > 0$ for $i \geq 0$, $q_i = p(i, i-1) > 0$ for $i \geq 1$, $q_0 = 0$, and $r_i = 1 - p_i - q_i = p(i, i) \geq 0$ for $i \geq 0$. This is an irreducible chain.

For $k \geq 0$, let

$$\varphi(k) = \sum_{m=0}^{k-1} \prod_{j=1}^{m} \frac{q_j}{p_j}.$$

Here, $\varphi(0) = 0$, $\varphi(1) = 1$. Let $N = \inf\{n : X_n = 0\}$. Let $Y_n = X_{n \wedge N}$. Note that Y_n is also a BD chain, but now $p_0 = 0$, i.e., 0 is absorbing.

Claim 1. $\varphi(Y_n)$ is a martingale wrt. $\mathcal{F}_n = \sigma\{Y_0, \ldots, Y_n\}$. For this, we need to check that $E[\varphi(Y_{n+1}) \mid Y_n = k] = \varphi(k)$.

For $k = 0$, this is clear. For $k \geq 1$, this works out to be:

$$q_k\varphi(k-1) + r_k\varphi(k) + p_k\varphi(k+1) = \varphi(k)$$
$$q_k\varphi(k-1) + p_k\varphi(k+1) = (p_k + q_k)\varphi(k)$$
$$\varphi(k+1) - \varphi(k) = \frac{q_k}{p_k}(\varphi(k) - \varphi(k-1)),$$

which is clearly true.

Now let

$$T_c = \inf\{n \geq 1 : X_n = c\}$$

Claim 2. For $0 \leq a < x < b$,

$$P_x(T_a > T_b) = \frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)}.$$

Let $T = T_a \wedge T_b$. Then $\varphi(X_{n \wedge T}) = \varphi(Y_{n \wedge T})$ is a bounded martingale, so u.i., and $P_x(T < \infty) = 1$, so by optional stopping,

$$\varphi(x) = E_x(\varphi(Y_T)) = \varphi(a)P_x(T_a < T_b) + \varphi(b)P_x(T_a > T_b).$$

Claim 3. 0 is recurrent iff

$$\varphi(\infty) = \sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{q_j}{p_j} = \infty.$$

We have from Claim 2, for any $x > 0$, and large enough b,

$$P_x(T_0 > T_b) = \frac{\varphi(x)}{\varphi(b)}.$$

By sending $b \to \infty$, we get

$$1 - \rho_{x0} = P_x(T_0 = \infty) = 0$$

iff $\varphi(b) \to \infty$.

For example if $p_i = p$, $q_i = q$, then $\varphi(\infty) = \infty$ if $q \geq p$ and $1/(1 - q/p)$ if $q < p$.