Markov chains: invariant measures

Janko Gravner

MAT 235B

March 9, 2024

Invariant measures

We are considering a Markov chain X_n on a countable state space S with transition probabilities $p(\cdot, \cdot)$.

Let $\mu : S \to [0, \infty)$ be arbitrary. We extend μ to a measure on S, so it is actually a function on 2^S. We call μ a *stationary* (or *invariant*) measure if

$$\sum_{x \in S} \mu(x) p(x, y) = \mu(y) \quad \text{for every } y \in S$$

So, μ is a left eigenvector for p, with eigenvalue 1.

If $\mu(S) = 1$, then μ is a *stationary* (or *invariant*) *distribution*. In this case, if $\mu_{X_0} = \mu$, then $\mu_{X_n} = \mu$ for all n.

Invariant measures

We call a measure μ *reversible* if

$$\mu(x)p(x,y) = \mu(y)p(y,x) \quad \text{for all } x, y \in S$$

A reversible measure is invariant:

$$\sum_x \mu(x)p(x,y) = \sum_x \mu(y)p(y,x) = \mu(y)$$

Why the name? If μ is a reversible distribution,

$$\begin{aligned} & P_\mu[X_n = y \mid X_{n+1} = x_{n+1}, \ldots, X_{n+k} = x_{n+k}] \\ & = \frac{\mu(y)p(y, x_{n+1})p(x_{n+1}, x_{n+2}), \ldots, p(x_{n+k-1}, x_{n+k})}{\mu(x_{n+1})p(x_{n+1}, x_{n+2}), \ldots, p(x_{n+k-1}, x_{n+k})} \\ & = p(x_{n+1}, y) \end{aligned}$$

Statistically, it is impossible to tell the direction of time.

Invariant measures

Example. If $p(x, y) = p(y, x)$ for all $x, y \in S$, then $\mu \equiv 1$ is reversible. SSRW on $\mathbb{Z}^d$ is such an example.

In general, $\mu \equiv 1$ is invariant if and only if the transition matrix $p(\cdot, \cdot)$ is doubly stochastic, $\sum_x p(x, y) = 1$. As an example, consider SRW on a discrete circle, going counterclockwise w.p. p and clockwise w.p. $1 - p$. This chain has a doubly stochastic transition matrix but is non-reversible if $p \neq 1/2$.

Example. For birth-death chain,

$$\mu(x) = \prod_{k=1}^{x} \frac{p_{k-1}}{q_k}$$

(where $\mu(0) = 1$) is reversible. For this, we need to show

$$\mu(x)p(x, x-1) = \mu(x-1)p(x-1, x), \quad x \geq 1$$
$$\mu(x)p(x, x+1) = \mu(x+1)p(x+1, x), \quad x \geq 0$$

These two say the same thing, and the second says:

$$\prod_{k=1}^{x} \frac{p_{k-1}}{q_k} \cdot p_x = \prod_{k=1}^{x+1} \frac{p_{k-1}}{q_k} \cdot q_{x+1},$$

which clearly holds.

Existence of invariant measures

Assume that $x \in S$ is a recurrent state, a *base point*. As before, let $T_x = \inf\{n \geq 1 : X_n = x\}$. Let μ be the expected number of visits to y before returning to x (now including time 0), that is,

$$\begin{aligned}\mu(y) &= E_x\left[\sum_{n=0}^{T_x-1} 1_{\{X_n=y\}}\right] \\ &= E_x\left[\sum_{n=0}^{\infty} 1_{\{X_n=y, T_x>n\}}\right] \\ &= \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)\end{aligned}$$

Observe that $\mu(x) = 1$, and that also that

$$\mu(y) = E_x\left[\sum_{n=1}^{T_x} 1_{\{X_n=y\}}\right] = \sum_{n=1}^{\infty} P_x(X_n = y, T_x \geq n)$$

Existence of invariant measures

Let $q_n(x, y) = P_x(X_n = y, T_x > n)$, so that

$$\mu(y) = \sum_{n=0}^{\infty} q_n(x, y)$$

Theorem (Existence)

As defined above, μ is an invariant measure.

Proof.

Note that $\mu(x) = 1$. For any $z \in S$,

$$\sum_y \mu(y)p(y, z) = \sum_{n=0}^{\infty} \sum_y p(y, z)q_n(x, y),$$

and we need to prove that this equals $\mu(z)$.

Existence of invariant measures

Proof, continued.

Case 1: $z \neq x$.

$$
\begin{aligned}
\sum_y p(y,z) q_n(x,y) &= \sum_y q_n(x,y) P_x[X_{n+1} = z \mid X_n = y] \\
&= \sum_y q_n(x,y) P_x[X_{n+1} = z \mid X_n = y, T_x > n] \\
&\qquad\qquad\qquad\qquad (\text{as } \{T_x > n\} \in \mathcal{F}_n) \\
&= \sum_y P_x[X_{n+1} = z, X_n = y, T_x > n] \\
&= P_x[X_{n+1} = z, T_x > n] \qquad (*) \\
&= P_x[X_{n+1} = z, T_x > n+1] \qquad (\text{as } z \neq x) \\
&= q_{n+1}(x,z)
\end{aligned}
$$

Existence of invariant measures

Proof, continued.

and

$$\begin{aligned}\sum_{n=0}^{\infty} q_{n+1}(x,z) &= \sum_{n=0}^{\infty} q_n(x,z) \qquad \text{(as } z \neq x,\ q_0(x,z) = 0\text{)} \\ &= \mu(z)\end{aligned}$$

Existence of invariant measures

Proof, continued.

Case 2: $z = x$. The above computation is valid through (*), where we join it.

$$\sum_y p(y,x)q_n(x,y) = P_x[X_{n+1} = x, T_x > n] \qquad (*)$$

$$= P_x[T_x = n+1]$$

and

$$\sum_{n=0}^{\infty} P_x[T_x = n+1] = P({}_x T_x < \infty) = 1 = \mu(x)$$

Existence of invariant measures

Proof, continued.

Finally, we need to prove that $\mu(y) < \infty$ for all y. If $\rho_{xy} = 0$, then $\mu(y) = 0$. If $\rho_{xy} > 0$, then $\rho_{yx} > 0$ and $p^n(y, x) > 0$ for some n. We already know that $\sum_z \mu(z)p^n(z, x) = 1$, and so $\mu(y)p^n(y, x) \leq 1$. □

Uniqueness of invariant measures

Theorem (Uniqueness)

Assume a MC is irreducible and recurrent. Then it has a unique stationary measure, up to constant multipliers.

Uniqueness of invariant measures

Proof.

Let ν be an arbitrary stationary measure. Pick $a \in S$ to use as a base point for μ. Then, for any $n \geq 1$,

$$\begin{aligned}
\nu(x) &= \sum_y \nu(y)p(y,x) = \nu(a)p(a,x) + \sum_{y \neq a} \nu(y)p(y,x) \\
&= \nu(a)P_a(X_1 = x) + \sum_{y \neq a} \sum_z \nu(z)p(z,y)p(y,x) \\
&= \nu(a)P_a(X_1 = x) + \sum_{y \neq a} \nu(a)p(a,y)p(y,x) \\
&\quad + \sum_{y \neq a, z \neq a} \nu(z)p(z,y)p(y,x)
\end{aligned}$$

Uniqueness of invariant measures

Proof.

$$
\begin{aligned}
\nu(x) &= \nu(a)P_a(X_1 = x) + \sum_{y\neq a} \nu(a)p(a,y)p(y,x) \\
&\qquad + \sum_{y\neq a, z\neq a} \nu(z)p(z,y)p(y,x) \\
&= \nu(a)P_a(X_1 = x) + \nu(a)P_a(X_1 \neq a, X_2 = x) \\
&\qquad + \sum_{y\neq a, z\neq a} \nu(a)p(a,z)p(z,y)p(y,x) + (\text{sth.} \geq 0) \\
&= \nu(a)P_a(X_1 = x) + \nu(a)P_a(X_1 \neq a, X_2 = x) \\
&\qquad + \nu(a)P_a(X_1 \neq a, X_2 \neq a, X_3 = x) + (\text{sth.} \geq 0) \\
&\dots \\
&= \nu(a)\sum_{m=1}^{n} P_a(X_1 \neq a, \dots, X_{m-1} \neq a, X_m = x) + (\text{sth.} \geq 0)
\end{aligned}
$$

Uniqueness of invariant measures

Proof, continued.

So,

$$\begin{aligned}\nu(x) &\geq \nu(a)\sum_{m=1}^{\infty} P_a(X_1 \neq a, \ldots, X_{m-1} \neq a, X_m = x) \\ &= \nu(a)\sum_{m=1}^{\infty} P_a[X_m = x, T_a \geq m] = \nu(a)\mu(x)\end{aligned}$$

Uniqueness of invariant measures

Proof, continued.

So, $\nu(x) \geq \nu(a)\mu(x)$, with equality when $x = a$. So, for any $n \geq 0$,

$$\nu(a) = \sum_x \nu(x)p^n(x,a) \geq \nu(a) \sum_x \mu(x)p^n(x,a)$$
$$= \nu(a)\mu(a) = \nu(a)$$

Take an $x \in S$. If, for some n, $p^n(x,a) > 0$, then $\nu(x) = \nu(a)\mu(x)$, to avoid the strict inequality above. By irreducibility, such n exists for any x. $\square$

Stationary distributions

Proposition

If there is a stationary distribution π, then all $y \in S$ with $\pi(y) > 0$ are recurrent.

Therefore, an invariant distribution cannot exist for irreducible transient chains, although an invariant measure may exist (e.g., 3d SSRW).

Stationary distributions

Proof.

Recall that $N(y)$ is the total number of visits to y, not counting time 0. If $\pi(y) > 0$,

$$E_\pi[N(y)] = \sum_x \pi(x) \sum_{n=1}^{\infty} p^n(x,y)$$
$$= \sum_{n=1}^{\infty} \sum_x \pi(x) p^n(x,y) = \sum_{n=1}^{\infty} \pi(y) = \infty$$

and

$$E_\pi[N(y)] = \sum_x \pi(x) E_x[N(y)] = \sum_x \pi(x) \frac{\rho_{xy}}{1-\rho_{yy}} \leq \frac{1}{1-\rho_{yy}},$$

so $\rho_{yy} = 1$. □

Stationary distributions

Proposition

Assume a chain is irreducible and has stationary distribution π. Then

$$\pi(x) = \frac{1}{E_x T_x} > 0,$$

for all $x \in S$.

Proof.

As, for all $x, y \in S$, $\pi(x)p^n(x, y) \leq \pi(y)$, irreducibility implies $\pi(x) > 0$ for all x and so all sites are recurrent. Therefore, π is unique. Moreover, using x as the base point,

$$\mu(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n) = \frac{\pi(y)}{\pi(x)}.$$

Sum over y: $E_x(T_x) = \sum_{n=0}^{\infty} P_x(T_x > n) = \frac{1}{\pi(x)}.$ □

Positive recurrence

A recurrent site x is *positive recurrent* if $E_x T_x < \infty$ and *null recurrent* otherwise.

Theorem

For an irreducible chain, TFAE:

(i) *some state is positive recurrent;*

(ii) *every state is positive recurrent; and*

(iii) *there exists a stationary distribution*

Proof.

The only remaining implication to prove is $(i) \implies (iii)$. To this end, assume $x \in S$ is a positive recurrent state. Then

$$\pi(y) = \frac{1}{E_x T_x} \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)$$

is a stationary measure and a distribution. □

Examples

Example. SSRW in dimensions 1 and 2 is null recurrent, as $\mu \equiv 1$ is invariant.

Example. Birth-death chain has $\mu(x) = \prod_{k=1}^{x} \frac{p_{k-1}}{q_k}$ and so is positive recurrent iff

$$\sum_{x=0}^{\infty} \mu(x) < \infty$$

We proved earlier that recurrence is equivalent to

$$\sum_{x=0}^{\infty} \frac{1}{\mu(x) p_x} = \infty.$$

Examples

Example. In M/G/1 queue, consider the embedded chain X_n with

$$p(0, k-1) = a_k,\ k \geq 2, \quad p(0,0) = a_0 + a_1,$$
$$p(j, j-1+k) = a_k,\ j \geq 1, k \geq 0,$$

$a_k > 0$ for all k. Let ξ be the r.v. with distribution $P(\xi = k) = a_k$ and $E\xi = \sum_{k=0}^{\infty} k a_k = \mu$.
We know this irreducible chain is recurrent iff $\mu \leq 1$.
Note that $E_1 T_0 = EN$, where N is the total population of the branching process with offspring distribution ξ. Also, $E_k T_{k-1} = EN$ for all $k \geq 1$, so $E_k T_0 = kEN$.

Examples

Let N_k be the population at time k, so that $N = \sum_{k=0}^{\infty} N_k$ and

$$EN = \sum_{k=0}^{\infty} EN_k = \sum_{k=0}^{\infty} \mu^k = \begin{cases} \frac{1}{1-\mu} & \mu < 1 \\ \infty & \mu \geq 1 \end{cases}$$

The chain is positive recurrent iff $\mu < 1$.

Can we compute the invariant distribution?

Examples

To begin with,

$$\begin{aligned} E_0 T_0 &= 1 + \sum_{k=1}^{\infty} a_k \frac{k-1}{1-\mu} \\ &= 1 + \frac{\mu - 1 + a_0}{1-\mu} \\ &= \frac{a_0}{1-\mu} \end{aligned}$$

We can compute π, as $\pi(0) = 1/E_0 T_0$, and then use recursion:

$$\begin{aligned} \pi(0) &= \pi(0)(a_0 + a_1) + \pi(1)a_0 \\ \pi(1) &= \pi(0)a_2 + \pi(1)a_1 + \pi(2)a_0 \\ &\cdots \end{aligned}$$