Math/Stat 235B, Winter 2024

Final Exam

Due: Wednesday, Mar. 20, by 4pm, in my office or in my mailbox, or in a single pdf file by email.

Directions: Work on these problems *alone*; you cannot discuss any part of the exam with anybody or use any books, papers, web sites, etc.; you can consult only your notes from 235AB. Give concise, but complete solutions and justify every statement you make. To facilitate grading, please solve each problem on a separate page, and put clearly labeled pages in proper order. I will not reply to specific questions about exam problems. However, if you think that a problem is misstated, please let me know. You will receive extra credit if you are the first person to spot a mistake and I will post any corrections or clarifications on the course's web page.

0. Report your homework score: award yourself a point for every homework problem which you essentially solved, by your own judgment.

1. Assume that X is a random variable with density f, and that $g : \mathbb{R} \to \mathbb{R}$ is a measurable function with $E|g(X)| < \infty$. Let $Y = 2|X| + X$. Find an expression for $E[g(X) \mid Y]$. In particular, give a formula for $E[X \mid Y]$ when X is Normal with expectation 1 and variance 1.

2. Let $U_0, U_1, \dots$ be i.i.d. random variables, uniformly distributed on $[-1, 1]$, $a_0 \geq a_1 \geq a_2 \geq \dots$ a sequence of strictly positive numbers, and $\xi_k = a_k U_k$. Let $S_n = \sum_{k=1}^n \xi_k$ and $R_n = \sum_{k=1}^n \xi_{k-1}\xi_k$.

(a) Show that both S_n and R_n are martingales.

(b) Determine $\langle S \rangle_n$ and $\langle R \rangle_n$.

(c) For each martingale in (a) determine the number α such that it converges a.s. iff $\sum_{k=1}^\infty a_k^\alpha < \infty$.

3. Assume that X_k are i.i.d. Exponential(1) r.v., i.e., they have density $e^{-x} 1_{[0,\infty)}$. Let $S_n = X_1 + \dots + X_n$. Fix a nonzero $a \in (-\infty, 1)$. Find a constant $b \in \mathbb{R}$ (depending on a) so that

$$M_n = e^{aS_n - bn}$$

is a martingale. Find the a.s. limit of M_n. Is the sequence M_n uniformly integrable?

4. Assume that p is the transitional probability matrix of an irreducible aperiodic Markov chain X_n with state space $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

(a) Prove that $P_0(\sup X_n = \infty) = 1$.

(b) Assume that $\sup_n E_0(X_n) < \infty$. Show that the chain is positive recurrent and that its invariant distribution π satisfies $\sum_{x=0}^\infty x\,\pi(x) < \infty$. (*Hint*. Find a lower bound on $P_0(X_n \leq M)$, for a suitably chosen constant M.)

We say that the chain has *bounded increments* if there exists a (deterministic) constant b so that $p(x, y) = 0$ for $|y - x| > b$.

(c) Assume that the chain has bounded increments. Moreover, assume that X_n is a submartingale (starting from any state). Prove that the chain is *not* positive recurrent. (*Hint*. One way is to find a lower bound on $P_x(T_0 \geq n)$ by stopping X_n when it exits $[1, bn]$.)

(d) Let $T = \inf\{n \geq 0 : X_n = 0\}$, and assume that $X_{n\wedge T}$ is a supermartingale (starting from any state). Show that the chain is recurrent.

(e) With the same T as in (d), assume now (throughout this part) that $X_{n\wedge T}$ is a martingale (starting from any state). Show that, if the chain also has bounded increments, then X_n is null recurrent. Give an example of such a chain. Show by an example that the chain may be positive recurrent if it does not have bounded increments.

5. Let ξ be a $\mathbb{Z}_+$–valued random variable. Define a Markov Chain on $\mathbb{Z}_+$ with transition probabilities

$$p(x, y) = P((x + 1 - \xi)_+ = y).$$

Assume that $P(\xi = 0) > 0$, $P(\xi > 1) > 0$. Let $\varphi(s) = E(s^\xi)$, $s \in (0, 1)$.
(a) Show that the Markov chain is irreducible and aperiodic.
(b) Assume $E\xi > 1$. Show that there exists an $s \in (0, 1)$ so that $s = \varphi(s)$ and $\mu(x) = s^x$ is an invariant measure.
(c) Assume $E\xi > 1$. Show that the chain is positive recurrent. Assuming $X_0 = x$, what is the limiting proportion of time the chain spends at 0, i.e., $\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} 1_{\{X_k=0\}}$?
(d) Assume $E\xi < 1$. Show that the chain is transient. (*Hint*. Compare X_n to the random walk $S_n = 1 - \xi_1 + \cdots + 1 - \xi_n$, where $\xi_i \overset{d}{=} \xi$ are i.i.d.)
(e) Assume $E\xi = 1$. Show that the chain is null recurrent. (*Hint*: revisit computation in (b) to find an invariant measure which is not summmable.)

6. Assume $a_n \in (0, 1)$, $n = 1, 2, \ldots$, and $p \in (0, 1]$. Assume the Markov Chain on $\mathbb{Z}_+ = \{0, 1, \ldots\}$ has transition probabilities: $p(n, n) = 1 - p$, $p(n, n + 1) = pa_n$, $p(n, 0) = p(1 - a_n)$ for $n \geq 1$; and $p(0, 1) = 1$.

(a) Show that the chain is irreducible and aperiodic. Show that $\lim_n P_0(X_n = 0)$ exists.

(b) Assume that $p = 1$. Show that the chain is recurrent if and only if $\sum_{n=1}^{\infty}(1 - a_n) = \infty$.

(c) Assume that $p = 1$. Assume that, for $n \geq 1$, $a_n = e^{-c/n}$ for some $c > 0$. Determine for which c is the chain null recurrent and for which c it is positive recurrent.

(d) Show that (b) holds and (c) has the same answer for all $p \in (0, 1]$.