
Homework Set No. 1, solution sketches.

Durrett, 1.1. If \( Durrett, 1.11. \) Compute now follows that for any \( P \) counterexample to (ii) to get you change your mind. Opening an empty room does not change the probabilities, as it introduces no

1. The probability that you get the prize if you stick by your original choice remains \( \frac{1}{3} \), and is \( \frac{2}{3} \) if your original choice is correct \( \frac{1}{3} \). A counterexample to (ii) is \( P((X, Y) = (1, 1)) = P((X, Y) = (0, 0)) = \frac{1}{3} \). A counterexample to (iii) is \( P((X, Y) = (1, 1)) = P((X, Y) = (0, 0)) = \frac{1}{3} \).

Durrett, 1.11. Compute \( E((X - Y)^2 | \mathcal{G}) \) and take the expectation.

Durrett, 1.12. Let \( X = E(Y | \mathcal{G}) \). By the assumption, \( X \neq Y \). By Jensen, \( |X| = |E(Y | \mathcal{G})| \leq E(|Y| | \mathcal{G}) \), but since \( E(|X|) = E(|Y|) \), this implies that \( |X| = E(|Y| | \mathcal{G}) \) a.s. Now multiply both sides by \( 1_{\{X \geq 0\}} \) to get \( X_1(X \geq 0) = E(|Y| 1_{\{X \geq 0\}} | \mathcal{G}) \) and \( X_1(X \geq 0) = E(Y 1_{\{X \geq 0\}} | \mathcal{G}) \). Subtract and take expectations to get \( E(|Y| - Y) 1_{\{X \geq 0\}} = 0 \) and so \( E(|Y| - Y) 1_{\{X \geq 0\}} = 0 \) a.s. which is equivalent to saying that \( P(Y < 0, X \geq 0) = 0 \), or \( 1_{\{X \geq 0\}} \leq 1_{\{Y \geq 0\}} \) a.s. Since \( P(X \geq 0) = P(Y \geq 0) \), \( 1_{\{X \geq 0\}} = 1_{\{Y \geq 0\}} \) a.s. It now follows that for any \( c \in \mathbb{Q} \), \( 1_{\{X \geq c\}} = 1_{\{Y \geq c\}} \) a.s. hence \( P(X \neq Y) = 0 \).

1. The probability that you get the prize if you stick by your original choice remains \( \frac{1}{3} \), and is \( \frac{2}{3} \) if you change your mind. Opening an empty room does not change the probabilities, as it introduces no new information. In other words, you would certainly change your mind if you were offered to exchange your original choice for both of the other two rooms, but this is exactly the choice you are being offered.

As some of you have pointed out, the probability does change after Monty makes his choice, if his strategy for opening empty rooms is known. For example, assume (without loss of generality) that you choose room A, and Monty’s strategy is to open room B, if possible, otherwise room C. If he opens door C, you know that B contains the prize. If he opens door B, you the prize is in A or C with equal probability. Of course, even in this case, the a-priori (before Monty’s choice is known) probability that your original choice is correct is \( \frac{1}{3} \).

Now assume that Monty opens the two doors at random. Let \( H_1 = \{ \text{your choice is correct} \} \), \( H_2 = \{ \text{your choice is incorrect} \} \), \( A = \{ \text{Monty opens the door to an empty room} \} \). Then \( P(H_1) = \frac{1}{3} \), \( P(H_2) = \frac{2}{3} \), \( P(A | H_1) = 1 \), \( P(A | H_2) = \frac{1}{2} \), so by the Bayes’ formula, \( P(H_1 | A) = \frac{1}{2} \).

2. \( f_{M_1, M_2}(x, y) = 2e^{-(x+y)}1_{x \leq y} \) and \( f_{M_2}(y) = 2(e^{-y} - e^{-2y}) \).

(b) \( (X_1, \ldots, X_{n-1}) \) has conditional density, conditioned on \( S_n = s \), equal to \( f(x_1, \ldots, x_{n-1}) = (n - 1)!s^{n-1}1_{x_1 + \ldots + x_{n-1} \leq s} \). \( S^{-1}_{n}(X_1, \ldots, X_{n-1}) \) has conditional density, conditioned on \( S_n = s \), equal to \( f(y_1, \ldots, y_{n-1}) = (n - 1)!1_{y_1 + \ldots + y_{n-1} \leq 1} \), so \( S^{-1}_{n}(X_1, \ldots, X_{n-1}) \) is independent of \( S_n \).

3. \( \varphi(x) = (f(x)g(x) + f(-x)g(-x))/(f(x) + f(-x)) \). Note that \( \varphi \) is even, i.e., \( \varphi(x) = \varphi(|x|) \). The answer is \( \varphi(X) \). To show this, assume \( g \geq 0 \) and take a bounded even function \( h \geq 0 \). What needs to be
shown is that

(1) \[
\int_{\mathbb{R}} \varphi(x)f(x)h(x) \, dx = \int_{\mathbb{R}} g(x)f(x)h(x) \, dx.
\]

To show (1), merely check that \( \varphi(x)f(x) - g(x)f(x) \) is odd.

(b) Same answer as (a), as \( \sigma(X^2) = \sigma([X]) \).

(c) Let now \( \varphi(x) = \sum_{k \in \mathbb{Z}} f(x + k)g(x + k)/\left(\sum_{k \in \mathbb{Z}} f(x + k)\right) \). This is periodic with period 1, so the putative answer \( \varphi(X) \) is a function of \( F(X) \). Again, we need to check (1), now for \( g \geq 0 \) and an arbitrary bounded periodic function \( h \geq 0 \). Denote \( b(x) = \sum_{k \in \mathbb{Z}} f(x + k) \) and note that \( b \) is periodic with period 1. Then

\[
\int_{\mathbb{R}} \varphi(x)f(x)h(x) \, dx = \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \frac{f(x + k)g(x + k)}{b(x)} f(x)h(x) \, dx \right)
\]

\[
= \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \frac{f(x)g(x)}{b(x)} f(x - k)h(x) \, dx \right)
\]

\[
= \int_{\mathbb{R}} \left( \sum_{k \in \mathbb{Z}} f(x - k) \frac{f(x)g(x)}{b(x)} h(x) \right) \, dx
\]

\[
= \int_{\mathbb{R}} b(x) \frac{f(x)g(x)}{b(x)} h(x) \, dx = \int_{\mathbb{R}} g(x)f(x)h(x) \, dx.
\]

The second equal sign is justified by the change of variable \( x + k \mapsto x \). Of course we have used Fubini (or MCT) numerous times.