Math 235B, Winter 2024

Homework 1

Durrett, 4.1.3, 4.1.5, 4.1.6, 4.1.7, 4.1.9, 4.1.10. (*Expanded hint for 4.1.10*: Use Jensen to prove $|X| = E(|Y| \mid \mathcal{G})$, conclude that $E((|Y| - Y)1_{\{X \geq 0\}}) = 0$, then that $1_{\{X \geq 0\}} = 1_{\{Y \geq 0\}}$ a.s., and finally replace 0 by arbitrary $c \in \mathbb{Q}$.)

1. Assume that $p \in (0, 1)$. Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli randoms variables with parameter p, that is, $p = P(X_k = 1) = 1 - P(X_k = 0)$. Let $N = \inf\{k : X_k = 1\}$. Compute $E[X_k \mid N]$.

2. A set $A \in \mathcal{G}$ is an *atom* of a σ-algebra $\mathcal{G}$ if, for every $B \in \mathcal{G}$ with $B \subset A$, either $P(B) = P(A)$ or $P(B) = 0$. Assume X is a random variable with finite expectation. Show that $E[X \mid \mathcal{G}]$ is a.s. constant on any atom of $\mathcal{G}$ and determine that constant. (First formulate formally what this means.)

3. For a given random variable X and $g : \mathbb{R} \to \mathbb{R}$, such that $E|g(X)| < \infty$, find the formula for $E(g(X) \mid X_+)$. In particular, if X is a standard Normal random variable, compute $E(X \mid X_+)$.

4. Assume that for two random variables X and Y with finite expectation we have $E[X \mid Y] = Y$ and $E[Y \mid X] = X$. Show that $X = Y$ a.s. (*Hint*. Show that, for every $x \in \mathbb{R}$,

$$E[(Y - X)1_{\{X > x, Y > x\}}] = E[(X - Y)1_{\{Y \leq x < X\}}] = E[(X - Y)1_{\{X \leq x < Y\}}].)$$