Math 235B, Winter 2024

Homework 2

Durrett, 4.2.2, 4.2.3 ($\vee$ denotes the maximum), 4.2.9.

1. Let X_n be a sequence of random variables and $\mathcal{F}_n$ a filtration, $n = 0, 1, \ldots$. For each of the following statements, determine, with proof, whether it is true or false.
(a) If $(X_n, \mathcal{F}_n)$ is a predictable martingale, then $P(X_n = X_0 \text{ for all } n) = 1$.
(b) If $(X_n, \mathcal{F}_n)$ is a submartingale and $EX_n = EX_0$ for all n, then $(X_n, \mathcal{F}_n)$ is a martingale.
(c) If $(X_n, \mathcal{F}_n)$ is a martingale and $(X_n^2, \mathcal{F}_n)$ is also a martingale, then $P(X_n = X_0 \text{ for all } n) = 1$.
(d) If $(X_n, \mathcal{F}_n)$ is a martingale, then $(X_{n+1}, \mathcal{F}_n)$ is a martingale.

2. Assume τ and σ are stopping times for a filtration $\mathcal{F}_n$. Prove that also $\sigma \wedge \tau$, $\sigma \vee \tau$ and $\sigma + \tau$ are stopping times for the same filtration.

3. Let ξ_i be independent with $E|\xi_i| < \infty$ and $E\xi_i = 0$. Let $S_n = \xi_1 + \cdots + \xi_n$. Prove the following:
(a) $ES_1^+ \leq ES_1^+ \leq \cdots \leq ES_n^+$.
(b) $P(S_j \geq -2ES_n^+) \geq 1/2$ for $j = 1, \ldots, n$.
(c) For every $a > 0$, $P(\max_{1 \leq j \leq n} S_j \geq a + 2ES_n^+) \leq 2P(S_n \geq a)$.
(d) If the distribution of ξ_i is symmetric, i.e., $\xi_i \stackrel{d}{=} -\xi_i$, then one can erase $2ES_n^+$ in (c).
(*Hints for (c) and (d).* Introduce the stopping time τ, the first time $S_j \geq a + 2ES_n^+$. Divide the event on the left according to the value of τ and use (b). The symmetric case should follow from your proof of (c); it is Theorem 5.2.7 in Durrett, which you are *not* allowed to use.)