Math 235B, Winter 2024

Homework 3

Durrett, 4.2.6, 4.2.8.

1. *Voter model.* Start with an undirected graph: let V be a finite set, and let E be a collection of undirected edges, that is, $E \subset \{\{x,y\} \subset V : x \neq y\}$. The state of the system at time $n = 0, 1, 2, \ldots$ is determined by a function $\zeta_n : V \to \{0,1\}$. Assume that $\zeta_0(x)$, $x \in V$, are independent and $P(\xi_0(x) = 1) = p \in [0,1]$.

At each time n, an *ordered* pair of neighboring sites $(x,y) \in V \times V$, $\{x,y\} \in E$, is chosen uniformly; moreover, all the choices are independent, and also independent of ζ_0. Then

$$\zeta_{n+1}(z) = \begin{cases} \zeta_n(z) & z \neq x, \\ \zeta_n(y) & z = x. \end{cases}$$

In words, if 0's and 1's are opinions of people residing at sites of V, at each time a random ordered pair (x,y) of neighbors is chosen and then x assumes the opinion of y.

(a) Let $X_n = |\{\zeta_n = 1\}|$ be the number of opinions 1 at time n. Show that X_n is a martingale.

(b) Assume that the graph (V,E) is connected. Show that, with probability 1, eventually a total consensus is reached, i.e. everybody holds the same opinion. Compute the probability that the final consensus is the opinion 1.