Math 235B, Winter 2024

Homework 4

Durrett, 4.4.10, 4.4.11 (*Hint*. Use Kronecker's lemma (Theorem 2.5.9 in Durrett). You do not need to prove this well-known result from analysis.)

1. Assume that $\xi_1, \xi_2, \ldots$ are independent, with $E\xi_n^2 \le n^{-p}$, for all n. Show that $\sum_{n=1}^{\infty} \xi_n$ converges if $p > 2$, but that this is not necessary true if $p = 2$.

2. Give an example of a martingale with bounded increments that converges with probability strictly between 0 and 1. (*Hint*. Consider $T_n = \sum_{k=1}^{n} \eta\xi_k$, where $\eta, \xi_1, \xi_2, \ldots$ are independent.)

3. For a branching process with offspring distribution with mean μ and variance $\sigma^2 \in (0, \infty)$, let M_n be its martingale. Compute its bracket $\langle M \rangle_n$. When does it converge?

4. *A time-dependent branching process*. Start with a single individual. For $n = 1, 2, \ldots$, any individual at time $n-1$ (independently of everything else) produces 2 successors at time n with probability p_n and otherwise dies, i.e. produces no successors. Let X_n be the number of individuals at time n. Assume that $p_n > 1/2$ are nonincreasing and converge to 1/2. (A careful definition of this process is part of the problem.)

(a) Show that $E(X_n) = \prod_{k=1}^{n}(2p_k)$ and $E(X_n^2) = E(X_n)^2 \cdot \left(1 + \sum_{k=1}^{n} \frac{2}{E(X_k)}(1 - p_k)\right)$, for $n = 1, 2, \ldots$

(b) Show that $M_n = X_n/E(X_n)$ is a martingale, which is bounded in L^2 if and only if $\sum_{n=0}^{\infty} \frac{1}{E(X_n)} < \infty$.

(c) Show that the process dies out a.s., that is, $P(X_n = 0 \text{ for some } n) = 1$ if $\sum_{n=1}^{\infty}(2p_n - 1) < \infty$. Show also that $P(X_n > 0 \text{ for all } n) > 0$ if $\sum_{n=1}^{\infty} \exp(-\alpha \sum_{k=1}^{n}(2p_k - 1)) < \infty$ for some $\alpha < 1$.

(d) Let $p_n = (1/2 + n^{-\gamma}) \wedge 1$. Determine for which γ the process dies out a.s.