Math 235B, Winter 2024

Homework 5

Durrett, 4.6.1.

1. Let $X_n = \prod_{k=1}^n \xi_k$, where ξ_k are independent, $\xi_k \geq 0$, $E\xi_k = 1$. Show first that X_n converges to an a.s. finite r.v. X_∞. Let $a_k = E\sqrt{\xi_k}$. Show that the following are equivalent:

(a) X_n are u.i.;

(b) X_n converges in L^1;

(c) $EX_\infty = 1$; and

(d) $\prod_{k=1}^\infty a_k > 0$.

(*Hint*. Consider the martingale $M_n = \Pi_{k=1}^n \frac{\sqrt{\xi_k}}{a_k}$. If (d) does not hold, convergence of M_n implies that $X_n \to 0$. If (d) does hold, show first that $a_k \leq 1$, that M_n converges in L^2, and then that M_n^2 is u.i.) This is known as Kakutani's theorem.

2. Let ξ_n, $n = 1, 2, \dots$ be i.i.d. random variables with $P(\xi_1 = 0) = P(\xi_1 = 1) = 1/2$. (Think of n as time.) Let T be the first time the complete segment 1010001101 appears in the sequence $(\xi_1, \xi_2, \xi_3, \dots)$. Compute ET with hints provided below.

Just before *every* time $n = 1, 2, \dots$ start a new betting scheme: bet \$1 on $\xi_n = 1$. If $\xi_n = 0$, you lose the \$1, and this scheme is over. If $\xi_n = 1$, you gain \$1, and you bet your entire capital of \$2 on $\xi_{n+1} = 0$. If you lose, you lose \$2 and quit; if you win, you invest your new capital of \$4 into betting on $\xi_{n+2} = 1$, and so on through the entire 1010001101 sequence. Note: several schemes may go on at the same time, because you start a new one even if some of the old schemes are still winning. For example, if $\xi_1 = 1$, then you put \$2 of your old scheme on $\xi_2 = 0$, but also \$1 of your new scheme on $\xi_2 = 1$.

Let X_n be the total capital (of all schemes) at (i.e., right after) time n; this does not involve \$1 bet of the new scheme at time $n + 1$. In particular, X_n might be 0. Also, define $X_0 = 0$.

(a) Show that $M_n = X_n - n$ is a martingale.

(b) Show that $M_{T\wedge n}$ is a uniformly integrable martingale. Compute ET.

(c) How does ET change if the distribution of ξ_1 changes to $P(\xi_1 = 0) = 1 - p$, $P(\xi_1 = 1) = p$, for some $p \in (0, 1)$?

3. Assume that X_n is a non-negative supermartingale and T a stopping time.

(a) Prove that $E(X_T 1_{\{T<\infty\}}) \leq E(X_0)$.

(b) Prove that, for each $c > 0$, $P(\sup_n X_n \geq c) \leq E(X_0)/c$.

4. You are on a starship in space at distance R_0 from the space station. Your control system is not functioning properly. All you can do is set a distance to travel. After that, your ship performs a "space-hop": it moves the chosen distance in a randomly chosen direction, with all 3-dimensional directions equally likely. If you manage to reach a distance at most r from the station, you can call for help.

Let R_n be the distance from the station after n space-hops and D_n the distance chosen for the n'th hop. Make the natural assumption that D_n is predictable.

(a) Show that, whatever strategy you choose, $1/R_n$ is a supermartingale. If your stategy is such that the distance you choose at time n is always at most R_{n-1}, then $1/R_n$ is a martingale. (*Hint*: Argue by symmetry that the problem reduces to computing $(1/(4\pi)) \int_{S^2} (a^2 + 2abx + b^2)^{-1/2} \, d\sigma(x, y, z)$, for $a, b > 0$, where σ is the surface measure on the sphere S^2. The integral equals $(a \wedge b)/ab$.)

(b) Conclude that for every strategy, $P(\text{your ship ever reaches help}) \leq r/R_0$. (*Hint*: Use the previous problem.)