Math 236A, Fall 2012.

Homework Assignment 1

Due: Oct. 12, 2012

1. Assume Ω is a finite set. A *partition* of Ω comprises a collection of nonempty disjoint sets whose union is Ω. A partition Π is a *refinement* of a partition Π' if every set in Π is a subset of a set in Π'.
(a) Prove that the set of all σ-algebras on Ω is in natural one to one correspondence with a set of partitions of Ω. Also, if the σ-algebras $\mathcal{F}_0$ and $\mathcal{F}_1$ correspond to respective partitions Π_0 and Π_1, then $\mathcal{F}_0 \subset \mathcal{F}_1$ if and only if Π_1 is a refinement of Π_0.
(b) Prove that a random variable is measurable with respect to a σ-algebra $\mathcal{F}$ if and only if it is constant on every set of the corresponding partition.
(c) Given a random variable on $(\Omega, 2^\Omega)$, and a sigma algebra $\mathcal{F}$, find a formula for $E(X|\mathcal{F})$ in terms of the corresponding partition.

2. Let X be a standard Normal random variable. Compute $E(X|X_+)$.

3. Assume that P and Q are equivalent probability measures on a space $(\Omega, \mathcal{F})$, and let $\Lambda = \frac{dQ}{dP}$. Let X be a random variable with $E_P|X| < \infty$ and $E_Q|X| < \infty$. Prove a Bayes' formula: for a sigma algebra $\mathcal{G} \subset \mathcal{F}$,

$$E_Q(X|\mathcal{G}) = \frac{E_P(X\Lambda|\mathcal{G})}{E_P(\Lambda|\mathcal{G})},$$

Q-a.s. (First show that all conditional expectations exists and that the denominator is a.s. nonzero.)

4. Assume that a discrete financial market is viable. The time interval is $t = 0, \ldots, T$.
(a) Let h be a contingent claim, for which a replicating strategy θ exists. Assume that h is $\mathcal{F}_t$ measurable for some $t < T$. Show that the discounted claim $e^{-r(T-t)}h$ is replicated by the same strategy at time t.
(b) Assume that the market is also complete. Show that the filtration must be generated by the (vector of) prices alone, i.e., $\mathcal{F}_t = \sigma(S_0, \ldots, S_t)$.

5. Assume that a discrete financial market is given by $S_t = (S_t^0, \ldots, S_t^d)$, $t = 0, \ldots, T$. Assume also that the market is viable and complete. As in class, let $\tilde{S}_t = \frac{1}{S_t^0} S_t$. Moreover, let $\bar{S}_t = \frac{1}{S_t^1} S_t$. Let $\tilde{Q}$ be the unique measure that makes $\tilde{S}_t$ a martingale. Denote

$$\Lambda_t = \frac{S_t^1 S_0^0}{S_t^0 S_0^1}.$$

(a) Show that Λ_t is a Q-martingale.
(b) Show that the unique measure $\bar{Q}$ that makes $\bar{S}_t$ a martingale is given by

$$\frac{d\bar{Q}}{d\tilde{Q}} = \Lambda_T.$$

(c) Show also that for any contingent claim X,

$$E_{\tilde{Q}}\left(X \cdot \frac{S_0^0}{S_T^0}\right) = E_{\bar{Q}}\left(X \cdot \frac{S_0^1}{S_T^1}\right).$$

This is known as the *Principle of numeraire equivalence*.

6. Assume a viable *binary* market (with one bond β_t and one stock S_t, in which the conditional distribution of S_{t+1} given S_t is concentrated on two values), with natural filtration. Recall that such market is automatically complete. Let Q be the risk-neutral measure. Assume that Y_t is a Q-martingale. Prove that Y_t is a martingale transform of $\tilde{S}_t$.

7. Assume the CRR model. Every trader's dream is the *Buy Low Sell High* option

$$H = \max\{S_0, \ldots, S_T\} - \min\{S_0, \ldots, S_T\}.$$

Write a program that will compute the price of H (which should be quite expensive) for an arbitrary value of parameters u, d, r, S_0, and T.

8. Assume the CRR model with parameters $r = \rho\frac{s}{N}$, $u = \exp(\rho\frac{s}{N} + \sigma\sqrt{\frac{s}{N}}$, $d = \exp(\rho\frac{s}{N} - \sigma\sqrt{\frac{s}{N}})$, and time interval $[0, N]$. Assume also $S_0 = x$. Show that, as $N \to \infty$, the (time 0) price of a contingent claim $\Phi(S_N)$ converges to

$$E(\Phi(xe^Y))$$

where Y is a normal random variable. In the process, compute expectation and variance of Y.

9. Assume that a discrete market has $r = 1$ and $d = 2$.
(1) Assume that $S_0 = (1, 1, 1)$ and that $T = 1$, i.e., this is a one time-period market. Let (S_1^1, S_1^2) be either $(2, 4)$ or $(1/2, 1/4)$. Determine the set of all arbitrage strategies. Is the market viable?
(2) Assume that $S_0 = (1, 10, 10)$ and that $T = 2$. Assume that the pair (S_1^1, S_1^2) of stocks at time 1 is $(11, 9)$, $(11, 10)$, or $(8, 11)$. From $t = 1$ to $t = 2$ the pair of stocks makes the following moves: from $(11, 9)$ to $(14, 8)$, $(10, 13)$, or $(10, 8)$; from $(11, 10)$ to $(12, 11)$, or $(10, 9)$; and from $(8, 11)$ to $(12, 5)$, $(10, 14)$, or $(6, 11)$. Is the market viable and complete? If so, determine the unique equivalent martingale measure.