Math 236A, Fall 2012.

Homework Assignment 4

Due: Nov. 9, 2012

1. Show that the following processes are martingales, and compute their variances.
   (a) $X_t = \int_0^t |B_s| dB_s$
   (b) $X_t = \int_0^t M_s dB_s$, where $M_s = \max\{B_u : 0 \leq u \leq s\}$.
   (c) $X_t = \int_0^t B_s/2 dB_s$.
   (d) $X_t = \int_0^t I_s dB_s$, where $I_s = \int_0^s B_u^2 du$.
   (e) $X_t = \int_0^t B_s \bar{B}_s dB_s$, where $\bar{B}_s$ is a Brownian motion independent of $B_s$.
   (f) $X_t = \int_0^t B_{\xi_s} dB_s$, where $\xi$ is a uniform random variable on $[0, 1]$, independent of $B_s$.

2. Consider $\int_0^t B_{s+1} dB_s$, $0 \leq t \leq 1$.
   (a) This is not an Itô integral. Why?
   (b) Still, one may hope that, as $n \to \infty$, the limit of the approximating sums $\sum_{i=0}^{n-1} B_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})$ exists in an appropriate sense. Show that the limit indeed exists (state in what sense), express it with genuine Itô integrals, and show that it results in a continuous process which is not a martingale.

3. The figure depicts a realization of $B_{k/n}$ (blue) and $\sum_{i=0}^{k-1} B_{i/n}(B_{(i+1)/n} - B_{i/n})$ (red) for a large $n$, and $0 \leq k/n \leq 200$ (plotted against $k/n$ and interpolated). Explain the presence of almost linear segments in the red curve and its much larger oscillations.

4. Compute the following stochastic integrals. The answers may contain Riemann integrals.
   (a) $\int_0^t e^s dB_s$
   (b) $\int_0^t B_s^2 dB_s$
   (c) $\int_0^t e^{B_s} dB_s$.
   (d) $\int_0^t s B_s dB_s$. 

![Stochastic Integral Graph](image-url)
5. Fix $a > 0$ and let $\tau$ be the first time that Brownian motion $B_t$ hits either $a$ or $-a$. Compute, with careful proof:

$$E \int_0^\tau B_s^2 \, ds.$$ 

(*Hint: Itô’s formula.*)