1. (a) Assume that \( f = f(t, x) \) is a \( C^{1,2}(\mathbb{R}^+ \times \mathbb{R}) \) such that \( \partial_t f + \frac{1}{2} \partial_{xx} f = 0 \). Show that \( f(t, B_t) \) is a local martingale.

(b) For \( f \) as in (a), let \( g(t, x) = \int_0^x f(t, z) \, dz - \frac{1}{2} \int_0^t \partial_x f(s, 0) \, ds \). Show that \( g(t, B_t) \) is also a local martingale.

(c) Let \( h_n \) be the Hermite polynomial of degree \( n \). (The first four are 1, \( x \), \( x^2 - 1 \), \( x^3 - 3x \).) Let \( f_n(t, x) = t^{n/2} h_n(x/\sqrt{t}) \) for \( n \geq 0 \). Show that \( f_n(t, B_t) \) is a martingale for every \( n \).

(d) Define the processes \( X^n_t \) recursively by \( X^n_0 \equiv 1 \) and \( X^n_{t+1} = \int_0^t X^n_s \, ds \). Find the connection between \( X^n_t \) and the martingales from (c).

2. Assume that \( B_1^1 \) and \( B_2^2 \) are independent Brownian motions. Consider \( X_t = \int_0^t B^1_s \, dB^1_s \) and \( X_t = \int_0^t B^2_s \, dB^1_s \). Then the two bracket processes \( \langle X \rangle_t \) and \( \langle Y \rangle_t \) are equal in distribution, but it is not even true that \( X_t \) and \( Y_t \) are equal in distribution for a fixed \( t > 0 \). Prove this.

3. What should be the quadratic variation process for a process given by \( dX_t = a \, dt + b_1 \, dB^1_t + b_2 \, dB^2_t \)? Give the reasoning for your formula although a complete proof is not necessary. In particular, compute the quadratic variation process in the case \( dX_t = B^1_t \, dB^1_t + |B^1_t| \, dB^2_t \).