Homework Assignment 7

Due: Dec. 10, 2012, noon. Please leave the assignment in my mailbox or slide it under the door of my office. This assignment covers the entire course and you are required to make a credible attempt at every problem.

1. Let $\tau$ be a stopping time, with $0 \leq \tau \leq 1$, for the Brownian motion $B_t$ and $I = \int_0^1 B_t \, dt$.

(a) Show (carefully!) that $E(B_t B_{\tau}) = E(\tau \wedge t)$ for each $t \in [0, 1]$.

(b) Find a formula for $E[(I - B_{\tau})^2]$, in terms of $E\tau$, $E\tau^2$.

(c) Find a stopping time $\tau \in [0, 1]$ that minimizes the expression in (a).

2. Assume that $h : [0, 1] \to \mathbb{R}$ is an absolutely continuous (deterministic) function with $L^2$ derivative.

(a) Show that for any $\epsilon > 0$, $P(|B_t| < \epsilon$ for all $t \in [0, 1]) > 0$.

(b) Assume that $h : [0, 1] \to \mathbb{R}$ is an absolutely continuous (deterministic) function with an $L^2$ derivative, and with $h(0) = 0$. Show that for any $\epsilon > 0$, $P(|B_t - h(t)| < \epsilon$ for all $t \in [0, 1]) > 0$.

(c) Let $h$ be as in (b). Show that, with probability 1, there exists an integer $n$ so that $|B_n| < \epsilon$ and $|B_{t+n} - B_n - h| < \epsilon$ for all $t \in [0, 1]$.

(d) “A two-dimensional Brownian motion will eventually write your name near the origin.” Formulate this precisely (with all necessary assumptions) and explain how it follows from (c).

3. Consider the two processes

$$X_t = \int_0^t \cos(s) \, dB^1_s \quad \text{and} \quad Y_t = \int_0^t \sin(s) \, dB^2_s.$$ 

(a) Assume first that $B^1_t = B^2_t = B_t$ is the same Brownian motion. For which times $t$ are the random variables $X_t$ and $Y_t$ independent?

(b) Assume now that $B^1_t$ and $B^2_t$ are independent Brownian motions. Compute the bracket $\langle Z \rangle_t$ of the process $Z_t = X_t + Y_t$. What can you say about the distribution of the process $Z_t$?

4. Find the unique solution to the SDE

$$dX_t = a \, dt + \sigma X_t \, dB_t$$

with $X_0 = 0$. Here, $a$ and $\sigma$ are fixed constants.

5. Consider logistic growth with multiplicative noise, that is, the SDE

$$dX_t = (\lambda X_t - X^2_t) \, dt + \sigma X_t \, dB_t.$$ 

Here, $\lambda > 0$ and $\sigma > 0$ are fixed constants. Find the unique solution for any deterministic $X_0 = x_0 > 0$ explicitly, by deriving the equation for $Y_t = 1/X_t$. What can you say about the behavior of $X_t$ as $t \to \infty$?
6. In the Black-Scholes model, consider the \textit{All or Nothing} option that pays one unit of cash at time \( T \) provided the stock exceeds the value \( K \) some time during the interval \([0, T]\); otherwise it pays 0. That is, its value at time \( T \) is

\[
X = 1_{\{\max_{t \in [0, T]} S_t \geq K\}}.
\]

Find the arbitrage price of \( X \) at time 0.