4. Mathematical Finance in Discrete Time

4.1 The Model

We will study so-called finite markets – i.e. discrete-time models of financial markets in which all relevant quantities take a finite number of values. Following the approach of Harrison and Pliska (1981) and Taqqu and Willinger (1987), it suffices, to illustrate the ideas, to work with a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a finite number $|\Omega|$ of points ω, each with positive probability: $\mathbb{P}(\{\omega\}) > 0$.

We specify a time horizon T, which is the terminal date for all economic activities considered. (For a simple option-pricing model the time horizon typically corresponds to the expiry date of the option.)

As before, we use a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^T$ consisting of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T$: we take $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the trivial σ-field, $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$ (here $\mathcal{P}(\Omega)$ is the power-set of Ω, the class of all $2^{|\Omega|}$ subsets of Ω: we need every possible subset, as they all – apart from the empty set – carry positive probability).

The financial market contains $d+1$ financial assets. The usual interpretation is to assume one risk-free asset (bond, bank account) labeled 0, and d risky assets (stocks, say) labeled 1 to d. While the reader may keep this interpretation as a mental picture, we prefer not to use it directly. The prices of the assets at time t are random variables, $S_0(t,\omega), S_1(t,\omega), \ldots, S_d(t,\omega)$ say, non-negative and $\mathcal{F}_t$-measurable (i.e. adapted: at time t, we know the prices $S_i(t)$). We write $S(t) = (S_0(t), S_1(t), \ldots, S_d(t))'$ for the vector of prices at time t. Hereafter we refer to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the set of trading dates, the price process S and the information structure $\mathbb{F}$, which is typically generated by the price process S, together as a securities market model.

It will be essential to assume that the price process of at least one asset follows a strictly positive process.

Definition 4.1.1. *A numéraire is a price process $(X(t))_{t=0}^T$ (a sequence of random variables), which is strictly positive for all $t \in \{0, 1, \ldots, T\}$.*

For the standard approach the risk-free bank account process is used as numéraire. In some applications, however, it is more convenient to use a security other than the bank account and we therefore just use S_0 without further specification as a numéraire. We furthermore take $S_0(0) = 1$ (that is, we reckon in units of the initial value of our numéraire), and define $\beta(t) := 1/S_0(t)$ as a discount factor.

A *trading strategy* (or *dynamic portfolio*) φ is a $\mathbb{R}^{d+1}$ vector stochastic process $\varphi = (\varphi(t))_{t=1}^T = ((\varphi_0(t,\omega), \varphi_1(t,\omega), \ldots, \varphi_d(t,\omega))')_{t=1}^T$ which is predictable (or previsible): each $\varphi_i(t)$ is $\mathcal{F}_{t-1}$-measurable for $t \geq 1$. Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t – to be determined on the basis of information available *before* time t; i.e. the investor selects his time t portfolio after observing the prices $S(t-1)$. However, the portfolio $\varphi(t)$ must be established before, and held until after, announcement of the prices $S(t)$. The components $\varphi_i(t)$ may assume negative as well as positive values, reflecting the fact that we allow short sales and assume that the assets are perfectly divisible.

Definition 4.1.2. *The value of the portfolio at time t is the scalar product*

$$V_\varphi(t) = \varphi(t)\cdot S(t) := \sum_{i=0}^{d} \varphi_i(t) S_i(t), \ (t = 1, 2, \ldots, T) \quad \text{and } V_\varphi(0) = \varphi(1)\cdot S(0).$$

The process $V_\varphi(t,\omega)$ is called the wealth or value process of the trading strategy φ.

The initial wealth $V_\varphi(0)$ is called the *initial investment* or *endowment* of the investor.

Now $\varphi(t) \cdot S(t-1)$ reflects the market value of the portfolio just after it has been established at time $t-1$, whereas $\varphi(t) \cdot S(t)$ is the value just after time t prices are observed, but before changes are made in the portfolio. Hence

$$\varphi(t) \cdot (S(t) - S(t-1)) = \varphi(t) \cdot \Delta S(t)$$

is the change in the market value due to changes in security prices which occur between time $t-1$ and t. This motivates:

Definition 4.1.3. *The gains process G_φ of a trading strategy φ is given by*

$$G_\varphi(t) := \sum_{\tau=1}^{t} \varphi(\tau) \cdot (S(\tau) - S(\tau-1)) = \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta S(\tau), \quad (t = 1, 2, \ldots, T).$$

Observe the – for now – formal similarity of the gains process G_φ from trading in S following a trading strategy φ to the martingale transform of S by φ.

Define $\tilde{S}(t) = (1, \beta(t)S_1(t), \ldots, \beta(t)S_d(t))'$, the vector of discounted prices, and consider the *discounted value process*

$$\tilde{V}_\varphi(t) = \beta(t)(\varphi(t) \cdot S(t)) = \varphi(t) \cdot \tilde{S}(t), \quad (t = 1, 2, \ldots, T)$$

and the *discounted gains process*

$$\tilde{G}_\varphi(t) := \sum_{\tau=1}^{t} \varphi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau - 1)) = \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta\tilde{S}(\tau), \quad (t = 1, 2, \dots, T).$$

Observe that the discounted gains process reflects the gains from trading with assets 1 to d only, which in case of the standard model (a bank account and d stocks) are the risky assets.

We will only consider special classes of trading strategies.

Definition 4.1.4. *The strategy φ is self-financing, $\varphi \in \Phi$, if*

$$\varphi(t) \cdot S(t) = \varphi(t+1) \cdot S(t) \quad (t = 1, 2, \dots, T-1). \tag{4.1}$$

Interpretation. When new prices $S(t)$ are quoted at time t, the investor adjusts his portfolio from $\varphi(t)$ to $\varphi(t+1)$, without bringing in or consuming any wealth. The following result (which is trivial in our current setting, but requires a little argument in continuous time) shows that renormalising security prices (i.e. changing the numéraire) has essentially no economic effects.

Proposition 4.1.1 (Numéraire Invariance). *Let $X(t)$ be a numéraire. A trading strategy φ is self-financing with respect to $S(t)$ if and only if φ is self-financing with respect to $X(t)^{-1}S(t)$.*

Proof. Since $X(t)$ is strictly positive for all $t = 0, 1, \dots, T$ we have the following equivalence, which implies the claim:

$$\begin{gathered} \varphi(t) \cdot S(t) = \varphi(t+1) \cdot S(t) \quad (t = 1, 2, \dots, T-1) \\ \Leftrightarrow \\ \varphi(t) \cdot X(t)^{-1}S(t) = \varphi(t+1) \cdot X(t)^{-1}S(t) \quad (t = 1, 2, \dots, T-1). \end{gathered}$$

$\square$

Corollary 4.1.1. *A trading strategy φ is self-financing with respect to $S(t)$ if and only if φ is self-financing with respect to $\tilde{S}(t)$.*

We now give a characterization of self-financing strategies in terms of the discounted processes.

Proposition 4.1.2. *A trading strategy φ belongs to Φ if and only if*

$$\tilde{V}_\varphi(t) = V_\varphi(0) + \tilde{G}_\varphi(t), \quad (t = 0, 1, \dots, T). \tag{4.2}$$

Proof. Assume $\varphi \in \Phi$. Then using the defining relation (4.1), the numéraire invariance theorem and the fact that $S_0(0) = 1$

$$\begin{aligned} V_\varphi(0) + \tilde{G}_\varphi(t) = & \quad \varphi(1) \cdot S(0) + \sum_{\tau=1}^{t} \varphi(\tau) \cdot (\tilde{S}(\tau) - \tilde{S}(\tau-1)) \\ = & \quad \varphi(1) \cdot \tilde{S}(0) + \varphi(t) \cdot \tilde{S}(t) \\ & + \sum_{\tau=1}^{t-1} (\varphi(\tau) - \varphi(\tau+1)) \cdot \tilde{S}(\tau) - \varphi(1) \cdot \tilde{S}(0) \\ = & \quad \varphi(t) \cdot \tilde{S}(t) = \tilde{V}_\varphi(t). \end{aligned}$$

Assume now that (4.2) holds true. By the numéraire invariance theorem it is enough to show the discounted version of relation (4.1). Summing up to $t = 2$ (4.2) is

$$\varphi(2) \cdot \tilde{S}(2) = \varphi(1) \cdot \tilde{S}(0) + \varphi(1) \cdot (\tilde{S}(1) - \tilde{S}(0)) + \varphi(2) \cdot (\tilde{S}(2) - \tilde{S}(1)).$$

Subtracting $\varphi(2) \cdot \tilde{S}(2)$ on both sides gives $\varphi(2) \cdot \tilde{S}(1) = \varphi(1) \cdot \tilde{S}(1)$, which is (4.1) for $t = 1$. Proceeding similarly – or by induction – we can show $\varphi(t) \cdot \tilde{S}(t) = \varphi(t+1) \cdot \tilde{S}(t)$ for $t = 2, \ldots, T-1$ as required. □

We are allowed to borrow (so $\varphi_0(t)$ may be negative) and sell short (so $\varphi_i(t)$ may be negative for $i = 1, \ldots, d$). So it is hardly surprising that if we decide what to do about the risky assets and fix an initial endowment, the numéraire will take care of itself, in the following sense.

Proposition 4.1.3. *If $(\varphi_1(t), \ldots, \varphi_d(t))'$ is predictable and V_0 is $\mathcal{F}_0$-measurable, there is a unique predictable process $(\varphi_0(t))_{t=1}^T$ such that $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)'$ is self-financing with initial value of the corresponding portfolio $V_\varphi(0) = V_0$.*

Proof. If φ is self-financing, then by Proposition 4.1.2,

$$\tilde{V}_\varphi(t) = V_0 + \tilde{G}_\varphi(t) = V_0 + \sum_{\tau=1}^{t} (\varphi_1(\tau) \Delta\tilde{S}_1(\tau) + \ldots + \varphi_d(\tau) \Delta\tilde{S}_d(\tau)).$$

On the other hand,

$$\tilde{V}_\varphi(t) = \varphi(t) \cdot \tilde{S}(t) = \varphi_0(t) + \varphi_1(t)\tilde{S}_1(t) + \ldots + \varphi_d(t)\tilde{S}_d(t).$$

Equate these:

$$\begin{aligned} \varphi_0(t) = V_0 + & \sum_{\tau=1}^{t} (\varphi_1(\tau) \Delta\tilde{S}_1(\tau) + \ldots + \varphi_d(\tau) \Delta\tilde{S}_d(\tau)) \\ & -(\varphi_1(t)\tilde{S}_1(t) + \ldots + \varphi_d(t)\tilde{S}_d(t)), \end{aligned}$$

which defines $\varphi_0(t)$ uniquely. The terms in $\tilde{S}_i(t)$ are

$$\varphi_i(t)\Delta\tilde{S}_i(t) - \varphi_i(t)\tilde{S}_i(t) = -\varphi_i(t)\tilde{S}_i(t-1),$$

which is $\mathcal{F}_{t-1}$-measurable. So

$$\varphi_0(t) = V_0 + \sum_{\tau=1}^{t-1}(\varphi_1(\tau)\Delta\tilde{S}_1(\tau) + \ldots + \varphi_d(\tau)\Delta\tilde{S}_d(\tau))$$
$$-(\varphi_1(t)S_1(t-1) + \ldots + \varphi_d(t)\tilde{S}_d(t-1)),$$

where as $\varphi_1, \ldots, \varphi_d$ are predictable, all terms on the right-hand side are $\mathcal{F}_{t-1}$-measurable, so φ_0 is predictable. □

Remark 4.1.1. Proposition 4.1.3 has a further important consequence: for defining a gains process $\tilde{G}_\varphi$ only the components $(\varphi_1(t), \ldots, \varphi_d(t))'$ are needed. If we require them to be predictable they correspond in a unique way (after fixing initial endowment) to a self-financing trading strategy. Thus for the discounted world predictable strategies and final cash-flows generated by them are all that matters.

We now turn to the modeling of derivative instruments in our current framework. This is done in the following fashion.

Definition 4.1.5. *A contingent claim X with maturity date T is an arbitrary $\mathcal{F}_T = \mathcal{F}$-measurable random variable (which is by the finiteness of the probability space bounded). We denote the class of all contingent claims by $L^0 = L^0(\Omega, \mathcal{F}, I\!P)$.*

The notation L^0 for contingent claims is motivated by them being simply random variables in our context (and by the functional-analytic spaces used later on).

A typical example of a contingent claim X is an option on some underlying asset S; then (e.g. for the case of a European call option with maturity date T and strike K) we have a functional relation $X = f(S)$ with some function f (e.g. $X = (S(T) - K)^+$). The general definition allows for more complicated relationships which are captured by the $\mathcal{F}_T$-measurability of X (recall that $\mathcal{F}_T$ is typically generated by the process S).

4.2 Existence of Equivalent Martingale Measures

4.2.1 The No-arbitrage Condition

The central principle in the single period example was the absence of arbitrage opportunities, i.e. the absence of investment strategies for making profits without any exposure to risk. As mentioned there this principle is central for any market model, and we now define the mathematical counterpart of this economic principle in our current setting.

Definition 4.2.1. *Let $\tilde{\Phi} \subset \Phi$ be a set of self-financing strategies. A strategy $\varphi \in \tilde{\Phi}$ is called an arbitrage opportunity or arbitrage strategy with respect to $\tilde{\Phi}$ if $\mathbb{P}\{V_\varphi(0) = 0\} = 1$, and the terminal wealth of φ satisfies*

$$\mathbb{P}\{V_\varphi(T) \geq 0\} = 1 \quad \textit{and} \quad \mathbb{P}\{V_\varphi(T) > 0\} > 0.$$

So an arbitrage opportunity is a self-financing strategy with zero initial value, which produces a non-negative final value with probability one and has a positive probability of a positive final value. Observe that arbitrage opportunities are always defined with respect to a certain class of trading strategies.

Definition 4.2.2. *We say that a security market $\mathcal{M}$ is arbitrage-free if there are no arbitrage opportunities in the class Φ of trading strategies.*

We will allow ourselves to use 'no-arbitrage' in place of 'arbitrage-free' when convenient.

We will use the following mental picture in analyzing the sample paths of the price processes. We observe a realization $S(t,\omega)$ of the price process $S(t)$. We want to know which sample point $\omega \in \Omega$ – or random outcome – we have. Information about ω is captured in the filtration $\mathbb{F} = \{\mathcal{F}_t\}$. In our current setting we can switch to the unique sequence of partitions $\{\mathcal{P}_t\}$ corresponding to the filtration $\{\mathcal{F}_t\}$. So at time t we know the set $A_t \in \mathcal{P}_t$ with $\omega \in A_t$. Now recall the structure of the subsequent partitions. A set $A \in \mathcal{P}_t$ is the disjoint union of sets $A_1, \ldots, A_K \in \mathcal{P}_{t+1}$. Since $S(u)$ is $\mathcal{F}_u$-measurable $S(t)$ is constant on A and $S(t+1)$ is constant on the A_k, $k = 1, \ldots, K$. So we can think of A as the time 0 state in a single-period model and each A_k corresponds to a state at time 1 in the single-period model. We can therefore think of a multi-period market model as a collection of consecutive single-period markets. What is the effect of a 'global' no-arbitrage condition on the single-period markets?

Lemma 4.2.1. *If the market model contains no arbitrage opportunities, then for all $t \in \{0, 1, \ldots, T-1\}$, for all self-financing trading strategies $\varphi \in \Phi$ and for any $A \in \mathcal{P}_t$, we have*

(i) $\mathbb{P}(\tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) \geq 0 | A) = 1 \Rightarrow \mathbb{P}(\tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) = 0 | A) = 1,$

(ii) $\mathbb{P}(\tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) \leq 0 | A) = 1 \Rightarrow \mathbb{P}(\tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) = 0 | A) = 1.$

Observe that the conditions in the lemma are just the defining conditions of an arbitrage opportunity from Definition 4.2.1. They are formulated for a single-period model from t to $t+1$ with respect to the available information $\omega \in A$. The economic meaning of this result answers the question raised above. No arbitrage 'globally' implies no arbitrage 'locally'. From this the idea of the proof is immediate. Any local trading strategy can be embedded in a global strategy for which we can use the global no-arbitrage condition.

Proof. We only prove (i) ((ii) is shown in a similar fashion). Fix $t \in \{0, \dots, T-1\}$ and $\varphi \in \Phi$. Suppose $I\!P(\tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) \geq 0|A) = 1$ for some $A \in \mathcal{P}_t$ and define a new trading strategy ψ for all times $u = 1, \dots, T$ as follows:

For $u \leq t:\ \psi(u) = 0$ ('do nothing before time t').
For $u = t+1:\ \psi(t+1) = 0$ if $\omega \notin A$, and

$$\psi_k(t+1,\omega) = \begin{cases} \varphi_k(t+1,\omega) & \text{if } \omega \in A \text{ and } k \in \{1,\dots,d\}, \\ \\ \varphi_0(t+1,\omega) - \tilde{V}_\varphi(t,\omega) & \text{if } \omega \in A \text{ and } k = 0. \end{cases}$$

(If ω happens to be in A at time t, follow strategy φ when dealing with the risky assets, but modify the holdings in the numéraire appropriately in order to compensate for doing nothing when $\omega \notin A$.)

For $u > t+1 : \psi_k(u) = 0$ for $k \in \{1, \dots, d\}$ and

$$\psi_0(u,\omega) = \begin{cases} \tilde{V}_\psi(t+1,\omega) & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

(Invest the amount $\tilde{V}_\psi(t+1)$ into the numéraire account if ω happens to be in A, otherwise do nothing.)

The next step now is to show that the strategy ψ is a self-financing trading strategy. By construction ψ is predictable, hence a trading strategy. For $\omega \notin A$ $\psi \equiv 0$, so we only have to consider $\omega \in A$. The relevant point in time is $t+1$. Recall that $\psi(t) = 0$, hence $\psi(t) \cdot \tilde{S}(t) = 0$. Now

$$\begin{aligned} \psi(t+1) \cdot \tilde{S}(t) &= (\varphi_0(t+1) - \tilde{V}_\varphi(t))\tilde{S}_0(t) + \sum_{k=1}^{d} \varphi_k(t+1)\tilde{S}_k(t) \\ &= \sum_{k=0}^{d} \varphi_k(t+1)\tilde{S}_k(t) - \tilde{V}_\varphi(t) \\ &= \varphi(t+1) \cdot \tilde{S}(t) - \tilde{V}_\varphi(t) = \varphi(t) \cdot \tilde{S}(t) - \tilde{V}_\varphi(t) = 0, \end{aligned}$$

using the fact that φ is self-financing. Since $\psi(u) \cdot \tilde{S}(u) = 0$ for $u \leq t$ we have $\psi(u+1) \cdot \tilde{S}(u) = \psi(u) \cdot \tilde{S}(u)$ for all $u \leq t$ (and for all $\omega \in \Omega$). When $u > t+1$ and $\omega \in A$ we only hold the numéraire asset (with constant discounted value equal to 1), so

$$\psi(u+1) \cdot \tilde{S}(u) = \tilde{V}_\psi(t+1) = \psi(u) \cdot \tilde{S}(u).$$

Therefore the strategy ψ is self-financing.

We now analyze the value process of ψ. Using our assumption $I\!P(\tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) \geq 0|A) = 1$ we see that for all $u \geq t+1$ and $\omega \in A$

$$\begin{aligned}\tilde{V}_\psi(u) &= \psi(u)\cdot\tilde{S}(u) = \psi(t+1)\cdot\tilde{S}(t+1)\\ &= (\varphi_0(t+1)-\tilde{V}_\varphi(t))\tilde{S}_0(t+1) + \sum_{k=1}^{d}\varphi_k(t+1)\tilde{S}_k(t+1)\\ &= \sum_{k=0}^{d}\varphi_k(t+1)\tilde{S}_k(t+1) - \tilde{V}_\varphi(t)\\ &= \tilde{V}_\varphi(t+1)-\tilde{V}_\varphi(t)\geq 0.\end{aligned}$$

Since $\tilde{V}_\psi(T) = 0$ on A^c ψ defines a self-financing trading strategy with $\tilde{V}_\psi(0) = 0$ and $\tilde{V}_\psi(T) \geq 0$. The assumption of an arbitrage-free market implies $\tilde{V}_\psi(T) = 0$ or

$$\begin{aligned}0 &= I\!P(\tilde{V}_\psi(T) > 0) = I\!P\left(\{\tilde{V}_\psi(T) > 0\}\cap A\right)\\ &= I\!P(\tilde{V}_\varphi(t+1)-\tilde{V}_\varphi(t) > 0|A)I\!P(A).\end{aligned}$$

Therefore $I\!P(\tilde{V}_\varphi(t+1)-\tilde{V}_\varphi(t) = 0|A) = 1$. □

The fundamental insight in the single-period example was the equivalence of the no-arbitrage condition and the existence of risk-neutral probabilities. For the multi-period case we now use the probabilistic machinery of Chapter 2 to establish the corresponding result.

Definition 4.2.3. *A probability measure $I\!P^*$ on $(\Omega, \mathcal{F}_T)$ equivalent to $I\!P$ is called a martingale measure for $\tilde{S}$ if the process $\tilde{S}$ follows a $I\!P^*$-martingale with respect to the filtration $I\!F$. We denote by $\mathcal{P}(\tilde{S})$ the class of equivalent martingale measures.*

Proposition 4.2.1. *Let $I\!P^*$ be an equivalent martingale measure ($I\!P^* \in \mathcal{P}(\tilde{S})$) and $\varphi \in \Phi$ any self-financing strategy. Then the wealth process $\tilde{V}_\varphi(t)$ is a $I\!P^*$-martingale with respect to the filtration $I\!F$.*

Proof. By the self-financing property of φ (compare Proposition 4.1.2, (4.2)), we have

$$\tilde{V}_\varphi(t) = V_\varphi(0) + \tilde{G}_\varphi(t) \quad (t = 0, 1, \ldots, T).$$

So

$$\tilde{V}_\varphi(t+1) - \tilde{V}_\varphi(t) = \tilde{G}_\varphi(t+1) - \tilde{G}_\varphi(t) = \varphi(t+1)\cdot(\tilde{S}(t+1)-\tilde{S}(t)).$$

So for $\varphi \in \Phi$, $\tilde{V}_\varphi(t)$ is the martingale transform of the $I\!P^*$ martingale $\tilde{S}$ by φ (see Theorem 3.4.1) and hence a $I\!P^*$ martingale itself. □

Observe that in our setting all processes are bounded, i.e. the martingale transform theorem is applicable without further restrictions. The next result is the key for the further development.

Proposition 4.2.2. *If an equivalent martingale measure exists – that is, if $\mathcal{P}(\tilde{S}) \neq \emptyset$ – then the market $\mathcal{M}$ is arbitrage-free.*

Proof. Assume such a $I\!P^*$ exists. For any self-financing strategy φ, we have as before

$$\tilde{V}_\varphi(t) = V_\varphi(0) + \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta\tilde{S}(\tau).$$

By Proposition 4.2.1, $\tilde{S}(t)$ a (vector) $I\!P^*$-martingale implies $\tilde{V}_\varphi(t)$ is a P^*-martingale. So the initial and final $I\!P^*$-expectations are the same,

$$I\!E^*(\tilde{V}_\varphi(T)) = I\!E^*(\tilde{V}_\varphi(0)).$$

If the strategy is an arbitrage opportunity its initial value – the right-hand side above – is zero. Therefore the left-hand side $I\!E^*(\tilde{V}_\varphi(T))$ is zero, but $\tilde{V}_\varphi(T) \geq 0$ (by definition). Also each $I\!P^*(\{\omega\}) > 0$ (by assumption, each $I\!P(\{\omega\}) > 0$, so by equivalence each $I\!P^*(\{\omega\}) > 0$). This and $\tilde{V}_\varphi(T) \geq 0$ force $\tilde{V}_\varphi(T) = 0$. So no arbitrage is possible. □

Proposition 4.2.3. *If the market $\mathcal{M}$ is arbitrage-free, then the class $\mathcal{P}(\tilde{S})$ of equivalent martingale measures is non-empty.*

Because of the fundamental nature of this result we will provide two proofs. The first proof is based on our previous observation that the 'global' no-arbitrage condition implies also no-arbitrage 'locally'. We therefore can combine single-period results to prove the multi-period claim. The second prove uses functional-analytic techniques (as does the corresponding proof in Chapter 1), i.e. a variant of the *Hahn-Banach theorem.*

First proof. From Lemma 4.2.1 we know that each of the underlying single-period market models is free of arbitrage. By the results in Chapter 1 this implies the existence of risk-neutral probabilities. That is, for each $t \in \{0, 1, \ldots, T-1\}$ and each $A \in \mathcal{P}_t$ there exists a probability measure $I\!P(t, A)$ such that each cell $A_i \subset A,\ i = 1, \ldots, K_A$ in the partition $\mathcal{P}_{t+1}$ has a positive probability mass and

$$\sum_{i=1}^{K_A} I\!P(t, A)(A_i) = 1.$$

Furthermore $I\!E_{I\!P(t,A)}(\tilde{S}(t+1)) = \tilde{S}(t)$ (where we restrict ourselves to $\omega \in A$). We can think of the probability measures $I\!P(t, A)$ as conditional risk-neutral probability measures given the event A occurred at time t. Now we can define a probability measure $I\!P^*$ on Ω by defining the probabilities of the simple events $\{\omega\}$ (observe that $\mathcal{F}_T = \mathcal{P}(\Omega)$, hence the final partition consists of all simple events). To each such $\{\omega\}$ there exists a single path from 0 to T and $I\!P^*$ is set equal to the product of the conditional probabilities along the path. By construction

$$\sum_{\omega \in \Omega} I\!P^*(\{\omega\}) = 1.$$

Since the conditional risk-neutral probabilities are greater than 0, $I\!P^*(\{\omega\}) > 0$ for each $\omega \in \Omega$ and $I\!P^*$ is an equivalent measure. The final step is to show that $I\!P^*$ is a martingale measure. We thus have to show $I\!E^*(\tilde{S}_k(t+1)|\mathcal{F}_t) = \tilde{S}_k(t)$ for any $k = 1, \ldots, d,\ t = 0, \ldots, T-1$. Now $\tilde{S}_k(t)$ is $\mathcal{F}_t$-measurable, and as any $A \in \mathcal{F}_t$ can be written as a union of $A' \in \mathcal{P}_t$ the claim follows from

$$\int_{A'} \tilde{S}_k(t+1) dI\!P^* = \int_{A'} \tilde{S}_k(t) dI\!P^*,$$

which is true by construction of $I\!P^*$. (Recall that we have $I\!E_{I\!P(A,t)}(\tilde{S}_k(t+1)) = I\!E_{I\!P(A,t)}(\tilde{S}_k(t))$.) □

For the second proof (for which we follow Schachermayer (2003)) we need some auxiliary observations.

Recall the definition of arbitrage, i.e. Definition 4.2.1, in our finite-dimensional setting: a self-financing trading strategy $\varphi \in \Phi$ is an arbitrage opportunity if $V_\varphi(0) = 0$, $V_\varphi(T, \omega) \geq 0\ \forall \omega \in \Omega$ and there exists an $\omega \in \Omega$ with $V_\varphi(T, \omega) > 0$.

Now call $L^0 = L^0(\Omega, \mathcal{F}, I\!P)$ the set of random variables on $(\Omega, \mathcal{F})$ and

$$L^0_{++}(\Omega, \mathcal{F}, I\!P) := \{X \in L^0 : X(\omega) \geq 0\ \forall \omega \in \Omega \text{ and } \exists\, \omega \in \Omega \text{ s. t. } X(\omega) > 0\}.$$

(Observe that L^0_{++} is a *cone* – closed under vector addition and multiplication by positive scalars.) Using L^0_{++} we can write the arbitrage condition more compactly as

$$V_\varphi(0) = \tilde{V}_\varphi(0) = 0 \quad \Rightarrow \quad \tilde{V}_\varphi(T) \notin L^0_{++}(\Omega, \mathcal{F}, I\!P)$$

for any self-financing strategy φ.

The next lemma formulates the arbitrage condition in terms of discounted gains processes. The important advantage in using this setting (rather than a setting in terms of value processes) is that we only have to assume predictability of a vector process $(\varphi_1, \ldots, \varphi_d)$. Recall Remark 4.1.1 and Proposition 4.1.3 here: we can choose a process φ_0 in such a way that the strategy $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)$ has zero initial value and is self-financing.

Lemma 4.2.2. *In an arbitrage-free market any predictable vector process $\varphi' = (\varphi_1, \ldots, \varphi_d)$ satisfies*

$$\tilde{G}_{\varphi'}(T) \notin L^0_{++}(\Omega, \mathcal{F}, I\!P).$$

(Observe the slight abuse of notation: for the value of the discounted gains process the zeroth component of a trading strategy doesn't matter. Hence we use the operator $\tilde{G}$ for d-dimensional vectors as well.)

Proof. By Proposition 4.1.3 there exists a unique predictable process $(\varphi_0(t))$ such that $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d)$ has zero initial value and is self-financing. Assume $\tilde{G}_{\varphi'}(T) \in L^0_{++}(\Omega, \mathcal{F}, I\!P)$. Then using Proposition 4.1.2,

$$V_\varphi(T) = \beta(T)^{-1}\tilde{V}_\varphi(T) = \beta(T)^{-1}(V_\varphi(0) + \tilde{G}_\varphi(T)) = \beta(T)^{-1}\tilde{G}_{\varphi'}(T),$$

which – as $\tilde{G}_{\varphi'} \in L^0_{++}$ – is nonnegative and positive somewhere with positive probability. This says that φ is an arbitrage opportunity with respect to Φ. This contradicts our assumption of no arbitrage, so we conclude $\tilde{G}_{\varphi'}(T) \notin L^0_{++}(\Omega, \mathcal{F}, I\!P)$ as required. □

We now define the space of contingent claims, i.e. random variables on $(\Omega, \mathcal{F})$, which an economic agent may replicate with zero initial investment by pursuing some predictable trading strategy φ.

Definition 4.2.4. *We call the subspace K of $L^0(\Omega, \mathcal{F}, I\!P)$ defined by*

$$K = \{X \in L^0(\Omega, \mathcal{F}, I\!P) : X = \tilde{G}_\varphi(T),\ \varphi \ \textit{predictable}\}$$

the set of contingent claims attainable at price 0.

We can now restate Lemma 4.2.2 in terms of spaces

A market is arbitrage-free if and only if

$$K \cap L^0_{++}(\Omega, \mathcal{F}, I\!P) = \emptyset. \tag{4.3}$$

Second proof of Proposition 4.2.3. Since our market model is finite we can use results from Euclidean geometry, in particular we can identify L^0 with $I\!R^{|\Omega|}$). By assumption we have (4.3), i.e. K and L^0_{++} do not intersect. So K does not meet the subset

$$D := \{X \in L^0_{++} : \sum_{\omega\in\Omega} X(\omega) = 1\}.$$

Now D is a compact convex set. By the separating hyperplane theorem, there is a vector $\lambda = (\lambda(\omega) : \omega \in \Omega)$ such that for all $X \in D$

$$\lambda \cdot X := \sum_{\omega\in\Omega} \lambda(\omega)X(\omega) > 0, \tag{4.4}$$

but for all $\tilde{G}_\varphi(T)$ in K,

$$\lambda \cdot \tilde{G}_\varphi(T) = \sum_{\omega\in\Omega} \lambda(\omega)\tilde{G}_\varphi(T)(\omega) = 0. \tag{4.5}$$

Choosing each $\omega \in \Omega$ successively and taking X to be 1 on this ω and zero elsewhere, (4.4) tells us that each $\lambda(\omega) > 0$. So

$$\mathbb{P}^*(\{\omega\}) := \frac{\lambda(\omega)}{\sum_{\omega' \in \Omega} \lambda(\omega')}$$

defines a probability measure equivalent to $\mathbb{P}$ (no non-empty null sets). With $\mathbb{E}^*$ as $\mathbb{P}^*$-expectation, (4.5) says that

$$\mathbb{E}^* \left(\tilde{G}_\varphi(T) \right) = 0,$$

i.e.

$$\mathbb{E}^* \left(\sum_{\tau=1}^{T} \varphi(\tau) \cdot \Delta \tilde{S}(\tau) \right) = 0.$$

In particular, choosing for each i to hold only stock i,

$$\mathbb{E}^* \left(\sum_{\tau=1}^{T} \varphi_i(\tau) \Delta \tilde{S}_i(\tau) \right) = 0 \quad (i = 1, \ldots, d).$$

Since this holds for any predictable φ (boundedness holds automatically as Ω is finite), the martingale transform lemma tells us that the discounted price processes $(\tilde{S}_i(t))$ are $\mathbb{P}^*$-martingales. □

Note. Our situation is finite-dimensional, so all we have used here is Euclidean geometry. We have a subspace, and a cone not meeting the subspace except at the origin. Take λ orthogonal to the subspace on the same side of the subspace as the cone. The separating hyperplane theorem holds also in infinite-dimensional situations, where it is a form of the Hahn-Banach theorem of functional analysis (Appendix C). For proofs, variants and background, see e.g. Bott (1942) and Valentine (1964).

We now combine Propositions 4.2.2 and 4.2.3 as a first central theorem in this chapter.

Theorem 4.2.1 (No-arbitrage Theorem). *The market $\mathcal{M}$ is arbitrage-free if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted d-dimensional asset price process $\tilde{S}$ is a $\mathbb{P}^*$-martingale.*

4.2.2 Risk-Neutral Pricing

We now turn to the main underlying question of this text, namely the pricing of contingent claims (i.e. financial derivatives). As in Chapter 1 the basic idea is to reproduce the cash flow of a contingent claim in terms of a portfolio of the underlying assets. On the other hand, the equivalence of the no-arbitrage condition and the existence of risk-neutral probability measures imply the possibility of using risk-neutral measures for pricing purposes. We will explore the relation of these two approaches in this subsection.

We say that a contingent claim is *attainable* if there exists a *replicating strategy* $\varphi \in \Phi$ such that

$$V_\varphi(T) = X.$$

So the replicating strategy generates the same time T cash-flow as does X. Working with discounted values (recall we use β as the discount factor) we find

$$\beta(T)X = \tilde{V}_\varphi(T) = V(0) + \tilde{G}_\varphi(T). \tag{4.6}$$

So the discounted value of a contingent claim is given by the initial cost of setting up a replication strategy and the gains from trading. In a highly efficient security market we expect that the law of one price holds true, that is for a specified cash-flow there exists only one price at any time instant. Otherwise arbitrageurs would use the opportunity to cash in a riskless profit (recall that a whole industry of hedge funds rely on such opportunities, also see the case of option mispricing at former NatWest Markets as an excellent example of how arbitrageurs exploit mispricing). So the no-arbitrage condition implies that for an attainable contingent claim its time t price must be given by the value (initial cost) of any replicating strategy (we say the claim is uniquely replicated in that case). This is the basic idea of the *arbitrage pricing theory.*

Let us investigate replicating strategies a bit further. The idea is to replicate a given cash-flow at a given point in time. Using a self-financing trading strategy the investor's wealth may go negative at time $t < T$, but he must be able to cover his debt at the final date. To avoid negative wealth the concept of admissible strategies is introduced. A self-financing trading strategy $\varphi \in \Phi$ is called *admissible* if $V_\varphi(t) \geq 0$ for each $t = 0, 1, \ldots, T$. We write Φ_a for the class of admissible trading strategies. The modeling assumption of admissible strategies reflects the economic fact that the broker should be protected from unbounded short sales. In our current setting all processes are bounded anyway, so this distinction is not really needed and we use self-financing strategies when addressing the mathematical aspects of the theory. (In fact one can show that a security market which is arbitrage-free with respect to Φ_a is also arbitrage-free with respect to Φ.)

We now return to the main question of the section: given a contingent claim X, i.e. a cash-flow at time T, how can we determine its value (price) at time $t < T$? For an attainable contingent claim this value should be given by the value of any replicating strategy at time t, i.e. there should be a unique value process (say $V_X(t)$) representing the time t value of the simple contingent claim X. The following proposition ensures that the value processes of replicating trading strategies coincide, thus proving the uniqueness of the value process.

Proposition 4.2.4. *Suppose the market $\mathcal{M}$ is arbitrage-free. Then any attainable contingent claim X is uniquely replicated in $\mathcal{M}$.*

Proof. Suppose there is an attainable contingent claim X and strategies φ and ψ such that

$$V_\varphi(T) = V_\psi(T) = X,$$

but there exists a $\tau < T$ such that

$$V_\varphi(u) = V_\psi(u) \text{ for every } u < \tau \text{ and } V_\varphi(\tau) \neq V_\psi(\tau).$$

Define $A := \{\omega \in \Omega : V_\varphi(\tau,\omega) > V_\psi(\tau,\omega)\}$, then $A \in \mathcal{F}_\tau$ and $I\!P(A) > 0$ (otherwise just rename the strategies). Define the $\mathcal{F}_\tau$-measurable random variable $Y := V_\varphi(\tau) - V_\psi(\tau)$ and consider the trading strategy ξ defined by

$$\xi(u) = \begin{cases} \varphi(u) - \psi(u), & u \le \tau \\ \mathbf{1}_{A^c}(\varphi(u) - \psi(u)) + \mathbf{1}_A(Y\beta(\tau), 0, \dots, 0), & \tau < u \le T. \end{cases}$$

The idea here is to use φ and ψ to construct a self-financing strategy with zero initial investment (hence use their difference ξ) and put any gains at time τ in the savings account (i.e. invest them risk-free) up to time T.

We need to show formally that ξ satisfies the conditions of an arbitrage opportunity. By construction ξ is predictable and the self-financing condition (4.1) is clearly true for $t \neq \tau$, and for $t = \tau$ we have using that $\varphi, \psi \in \Phi$

$$\begin{aligned} \xi(\tau) \cdot S(\tau) &= (\varphi(\tau) - \psi(\tau)) \cdot S(\tau) = V_\varphi(\tau) - V_\psi(\tau), \\ \xi(\tau+1) \cdot S(\tau) &= \mathbf{1}_{A^c}(\varphi(\tau+1) - \psi(\tau+1)) \cdot S(\tau) + \mathbf{1}_A Y\beta(\tau) S_0(\tau) \\ &= \mathbf{1}_{A^c}(\varphi(\tau) - \psi(\tau)) \cdot S(\tau) + \mathbf{1}_A(V_\varphi(\tau) - V_\psi(\tau))\beta(\tau)\beta^{-1}(\tau) \\ &= V_\varphi(\tau) - V_\psi(\tau). \end{aligned}$$

Comparing these two, ξ is self-financing, and its initial value is zero. Also

$$V_\xi(T) = \mathbf{1}_{A^c}(\varphi(T) - \psi(T)) \cdot S(T) + \mathbf{1}_A(Y\beta(\tau), 0, \dots, 0) \cdot S(T).$$

The first term is zero, as $V_\varphi(T) = V_\psi(T)$. The second term is

$$\mathbf{1}_A Y\beta(\tau) S_0(T) \ge 0,$$

as $Y > 0$ on A, and indeed

$$I\!P\{V_\xi(T) > 0\} = I\!P\{A\} > 0.$$

Hence the market contains an arbitrage opportunity with respect to the class Φ of self-financing strategies. But this contradicts the assumption that the market $\mathcal{M}$ is arbitrage-free. $\square$

This uniqueness property allows us now to define the important concept of an arbitrage price process.

Definition 4.2.5. *Suppose the market is arbitrage-free. Let X be any attainable contingent claim with time T maturity. Then the arbitrage price process $\pi_X(t),\ 0 \le t \le T$ or simply arbitrage price of X is given by the value process of any replicating strategy φ for X.*

The construction of hedging strategies that replicate the outcome of a contingent claim (for example a European option) is an important problem in both practical and theoretical applications. Hedging is central to the theory of option pricing. The classical arbitrage valuation models, such as the Black-Scholes model Black and Scholes (1973), depend on the idea that an option can be perfectly hedged using the underlying asset (in our case the assets of the market model), so making it possible to create a portfolio that replicates the option exactly. Hedging is also widely used to reduce risk, and the kinds of delta-hedging strategies implicit in the Black-Scholes model are used by participants in option markets. We will come back to hedging problems subsequently.

Analyzing the arbitrage-pricing approach we observe that the derivation of the price of a contingent claim doesn't require any specific preferences of the agents other than nonsatiation, i.e. agents prefer more to less, which rules out arbitrage. So, the pricing formula for any attainable contingent claim must be independent of all preferences that do not admit arbitrage. In particular, an economy of risk-neutral investors must price a contingent claim in the same manner. This fundamental insight, due to Cox and Ross (1976) in the case of a simple economy – a riskless asset and one risky asset – and in its general form due to Harrison and Kreps (1979), simplifies the pricing formula enormously. In its general form the price of an attainable simple contingent claim is just the expected value of the discounted payoff with respect to an equivalent martingale measure.

Proposition 4.2.5. *The arbitrage price process of any attainable contingent claim X is given by the* **risk-neutral valuation formula**

$$\pi_X(t) = \beta(t)^{-1} \mathbb{E}^* \left(X\beta(T) | \mathcal{F}_t \right) \quad \forall t = 0, 1, \ldots, T, \tag{4.7}$$

where $\mathbb{E}^*$ is the expectation operator with respect to an equivalent martingale measure $\mathbb{P}^*$.

Proof. Since we assume the the market is arbitrage-free, there exists (at least) an equivalent martingale measure $\mathbb{P}^*$. By Proposition 4.2.1 the discounted value process $\tilde{V}_\varphi$ of any self-financing strategy φ is a $\mathbb{P}^*$-martingale. So for any contingent claim X with maturity T and any replicating trading strategy $\varphi \in \Phi$ we have for each $t = 0, 1, \ldots, T$

$$\begin{aligned}
\pi_X(t) &= V_\varphi(t) = \beta(t)^{-1} \tilde{V}_\varphi(t) \\
&= \beta(t)^{-1} E^* (\tilde{V}_\varphi(T) | \mathcal{F}_t) && \text{(as } \tilde{V}_\varphi(t) \text{ is a } \mathbb{P}^*\text{-martingale)} \\
&= \beta(t)^{-1} E^* (\beta(T) V_\varphi(T) | \mathcal{F}_t) && \text{(undoing the discounting)} \\
&= \beta(t)^{-1} E^* (\beta(T) X | \mathcal{F}_t) && \text{(as } \varphi \text{ is a replicating strategy for } X\text{).}
\end{aligned}$$

□

4.3 Complete Markets: Uniqueness of Equivalent Martingale Measures

The last section made clear that attainable contingent claims can be priced using an equivalent martingale measure. In this section we will discuss the question of the circumstances under which all contingent claims are attainable. This would be a very desirable property of the market $\mathcal{M}$, because we would then have solved the pricing question (at least for contingent claims) completely. Since contingent claims are merely $\mathcal{F}_T$-measurable random variables in our setting, it should be no surprise that we can give a criterion in terms of probability measures. We start with:

Definition 4.3.1. *A market $\mathcal{M}$ is complete if every contingent claim is attainable, i.e. for every $\mathcal{F}_T$-measurable random variable $X \in L^0$ there exists a replicating self-financing strategy $\varphi \in \Phi$ such that $V_\varphi(T) = X$.*

In the case of an arbitrage-free market $\mathcal{M}$ one can even insist on replicating nonnegative contingent claims by an admissible strategy $\varphi \in \Phi_a$. Indeed, if φ is self-financing and $I\!P^*$ is an equivalent martingale measure under which discounted prices $\tilde{S}$ are $I\!P^*$-martingales (such $I\!P^*$ exist since $\mathcal{M}$ is arbitrage-free and we can hence use the no-arbitrage theorem (Theorem 4.2.1)), $\tilde{V}_\varphi(t)$ is also a $I\!P^*$-martingale, being the martingale transform of the martingale $\tilde{S}$ by φ (see Proposition 4.2.1). So

$$\tilde{V}_\varphi(t) = E^*(\tilde{V}_\varphi(T)|\mathcal{F}_t) \quad (t = 0, 1, \dots, T).$$

If φ replicates X, $V_\varphi(T) = X \geq 0$, so discounting, $\tilde{V}_\varphi(T) \geq 0$, so the above equation gives $\tilde{V}_\varphi(t) \geq 0$ for each t. Thus all the values at each time t are non-negative – not just the final value at time T – so φ is admissible.

Theorem 4.3.1 (Completeness Theorem). *An arbitrage-free market $\mathcal{M}$ is complete if and only if there exists a unique probability measure $I\!P^*$ equivalent to $I\!P$ under which discounted asset prices are martingales.*

Proof. '$\Rightarrow$': Assume that the arbitrage-free market $\mathcal{M}$ is complete. Then for any $\mathcal{F}_T$-measurable random variable X (contingent claim), there exists an admissible (so self-financing) strategy φ replicating X: $X = V_\varphi(T)$. As φ is self-financing, by Proposition 4.1.2,

$$\beta(T)X = \tilde{V}_\varphi(T) = V_\varphi(0) + \sum_{\tau=1}^{T} \varphi(\tau) \cdot \Delta\tilde{S}(\tau).$$

We know by the no-arbitrage theorem (Theorem 4.2.1), that an equivalent martingale measure $I\!P^*$ exists; we have to prove uniqueness. So, let $I\!P_1, I\!P_2$ be two such equivalent martingale measures. For $i = 1, 2$, $(\tilde{V}_\varphi(t))_{t=0}^T$ is a $I\!P_i$-martingale. So,

$$\mathbb{E}_i(\tilde{V}_\varphi(T)) = \mathbb{E}_i(\tilde{V}_\varphi(0)) = V_\varphi(0),$$

as the value at time zero is non-random ($\mathcal{F}_0 = \{\emptyset, \Omega\}$) and $\beta(0) = 1$. So

$$\mathbb{E}_1(\beta(T)X) = \mathbb{E}_2(\beta(T)X).$$

Since X is arbitrary, $\mathbb{E}_1, \mathbb{E}_2$ have to agree on integrating all integrands. Now $\mathbb{E}_i$ is expectation (i.e. integration) with respect to the measure $\mathbb{P}_i$, and measures that agree on integrating all integrands must coincide. So $\mathbb{P}_1 = \mathbb{P}_2$, giving uniqueness as required.

'$\Leftarrow$': Assume that the arbitrage-free market $\mathcal{M}$ is incomplete: then there exists a non-attainable $\mathcal{F}_T$-measurable random variable X (a contingent claim). By Proposition 4.1.3, we may confine attention to the risky assets $S_1, \dots, S_d$, as these suffice to tell us how to handle the numéraire S_0.

Consider the following set of random variables:

$$\tilde{K} := \left\{ Y \in L^0 : Y = Y_0 + \sum_{t=1}^{T} \varphi(t) \cdot \Delta \tilde{S}(t),\ Y_0 \in \mathbb{R}\ ,\ \varphi \ \text{ predictable} \right\}.$$

(Recall that Y_0 is $\mathcal{F}_0$-measurable and set $\varphi = ((\varphi_1(t), \dots, \varphi_d(t))')_{t=1}^T$ with predictable components.) Then by the above reasoning, the discounted value $\beta(T)X$ does not belong to $\tilde{K}$, so $\tilde{K}$ is a *proper* subset of the set L^0 of all random variables on Ω (which may be identified with $\mathbb{R}^{|\Omega|}$). Let $\mathbb{P}^*$ be a probability measure equivalent to $\mathbb{P}$ under which discounted prices are martingales (such $\mathbb{P}^*$ exist by the no-arbitrage theorem (Theorem 4.2.1). Define the scalar product

$$(Z, Y) \to \mathbb{E}^*(ZY)$$

on random variables on Ω. Since $\tilde{K}$ is a proper subset, there exists a non-zero random variable Z orthogonal to $\tilde{K}$ (since Ω is finite, $\mathbb{R}^{|\Omega|}$ is Euclidean: this is just Euclidean geometry). That is,

$$\mathbb{E}^*(ZY) = 0, \quad \forall\, Y \in \tilde{K}.$$

Choosing the special $Y = 1 \in \tilde{K}$ given by $\varphi_i(t) = 0, t = 1, 2, \dots, T;\ i = 1, \dots, d$ and $Y_0 = 1$ we find

$$\mathbb{E}^*(Z) = 0.$$

Write $\|X\|_\infty := \sup\{|X(\omega)| : \omega \in \Omega\}$, and define $\mathbb{P}^{**}$ by

$$\mathbb{P}^{**}(\{\omega\}) = \left(1 + \frac{Z(\omega)}{2\,\|Z\|_\infty}\right) \mathbb{P}^*(\{\omega\}).$$

By construction, $\mathbb{P}^{**}$ is equivalent to $\mathbb{P}^*$ (same null sets - actually, as $\mathbb{P}^* \sim \mathbb{P}$ and $\mathbb{P}$ has no non-empty null sets, neither do $\mathbb{P}^*, \mathbb{P}^{**}$). From $\mathbb{E}^*(Z) = 0$, we see that $\sum \mathbb{P}^{**}(\omega) = 1$, i.e. is a probability measure. As Z is non-zero, $\mathbb{P}^{**}$ and $\mathbb{P}^*$ are *different*. Now

$$\mathbb{E}^{**}\left(\sum_{t=1}^{T}\varphi(t)\cdot\Delta\tilde{S}(t)\right) = \sum_{\omega\in\Omega}\mathbb{P}^{**}(\omega)\left(\sum_{t=1}^{T}\varphi(t,\omega)\cdot\Delta\tilde{S}(t,\omega)\right)$$
$$= \sum_{\omega\in\Omega}\left(1+\frac{Z(\omega)}{2\,\|Z\|_\infty}\right)\mathbb{P}^*(\omega)\left(\sum_{t=1}^{T}\varphi(t,\omega)\cdot\Delta\tilde{S}(t,\omega)\right).$$

The '1' term on the right gives

$$\mathbb{E}^*\left(\sum_{t=1}^{T}\varphi(t)\cdot\Delta\tilde{S}(t)\right),$$

which is zero since this is a martingale transform of the $\mathbb{P}^*$-martingale $\tilde{S}(t)$ (recall martingale transforms are by definition null at zero). The 'Z' term gives a multiple of the inner product

$$(Z, \sum_{t=1}^{T}\varphi(t)\cdot\Delta\tilde{S}(t)),$$

which is zero as Z is orthogonal to $\tilde{K}$ and $\sum_{t=1}^{T}\varphi(t)\cdot\Delta\tilde{S}(t)\in\tilde{K}$. By the martingale transform lemma (Lemma 3.4.1), $\tilde{S}(t)$ is a $\mathbb{P}^{**}$-martingale since φ is an arbitrary predictable process. Thus $\mathbb{P}^{**}$ is a second equivalent martingale measure, different from $\mathbb{P}^*$. So incompleteness implies non-uniqueness of equivalent martingale measures, as required. □

Martingale Representation. To say that every contingent claim can be replicated means that every $\mathbb{P}^*$-martingale (where $\mathbb{P}^*$ is the risk-neutral measure, which is unique) can be written, or *represented*, as a martingale transform (of the discounted prices) by a replicating (perfect-hedge) trading strategy φ. In stochastic-process language, this says that all $\mathbb{P}^*$-martingales can be *represented* as martingale transforms of discounted prices. Such martingale representation theorems hold much more generally, and are very important. For background, see Revuz and Yor (1991) and Yor (1978).

4.4 The Fundamental Theorem of Asset Pricing: Risk-Neutral Valuation

We summarize what we have achieved so far. We call a measure $\mathbb{P}^*$ under which discounted prices $\tilde{S}(t)$ are $\mathbb{P}^*$-martingales a *martingale measure*. Such a $\mathbb{P}^*$ equivalent to the actual probability measure P is called an *equivalent martingale measure*. Then:

- *No-arbitrage theorem (Theorem 4.2.1):* If the market is *arbitrage-free*, equivalent martingale measures $I\!P^*$ *exist*.
- *Completeness theorem (Theorem 4.3.1):* If the market is *complete* (all contingent claims can be replicated), equivalent martingale measures are *unique*.

Combining:

Theorem 4.4.1 (Fundamental Theorem of Asset Pricing). *In an arbitrage-free complete market $\mathcal{M}$, there exists a unique equivalent martingale measure $I\!P^*$.*

The term *fundamental theorem of asset pricing* was introduced in Dybvig and Ross (1987). It is used for theorems establishing the equivalence of an economic modeling condition such as no-arbitrage to the existence of the mathematical modeling condition existence of equivalent martingale measures.

Assume now that $\mathcal{M}$ is an arbitrage-free complete market and let X be any contingent claim, φ a self-financing strategy replicating it (which exists by completeness), then:

$$V_\varphi(T) = X.$$

As $\tilde{V}_\varphi(t)$ is the martingale transform of the $I\!P^*$-martingale $\tilde{S}(t)$ (by $\varphi(t)$), $\tilde{V}_\varphi(t)$ is a $I\!P^*$-martingale. So $V_\varphi(0)(= \tilde{V}_\varphi(0)) = I\!E^*(\tilde{V}_\varphi(T)) = I\!E^*(\beta(T)X)$, giving us the risk-neutral pricing formula

$$V_\varphi(0) = I\!E^*(\beta(T)X).$$

More generally, the same argument gives $\tilde{V}_\varphi(t) = \beta(t)V_\varphi(t) = I\!E^*(\beta(T)X|\mathcal{F}_t)$:

$$V_\varphi(t) = \beta(t)^{-1} I\!E^*(\beta(T)X|\mathcal{F}_t) \quad (t = 0, 1, \dots, T). \tag{4.8}$$

It is natural to call $V_\varphi(0) = \pi_X(0)$ above the *arbitrage price* (or more exactly, *arbitrage-free price*) of the contingent claim X at time 0, and $V_X(t) = \pi_X(t)$ above the *arbitrage price* (or more exactly, *arbitrage-free price*) of the simple contingent claim X at time t. For, if an investor *sells* the claim X at time t for $V_X(t)$, he can follow strategy φ to replicate X at time T and clear the claim; an investor selling for this value is *perfectly hedged*. To sell the claim for *any other amount* would provide an arbitrage opportunity (as with the argument for put-call parity). We note that, to calculate prices as above, we need to know only:

1. Ω, the set of all possible states,
2. the σ-field $\mathcal{F}$ and the filtration (or information flow) $(\mathcal{F}_t)$,
3. $I\!P^*$.

We do **not** need to know the underlying probability measure $I\!P$ – only its null sets, to know what 'equivalent to $I\!P$' means (actually, in this finite model, there are no non-empty null-sets, so we do not need to know even this).

Now pricing of contingent claims is our central task, and for pricing purposes $\mathbb{P}^*$ is vital and $\mathbb{P}$ itself irrelevant. We thus may – and shall – focus attention on $\mathbb{P}^*$, which is called the *risk-neutral* probability measure. *Risk-neutrality* is the central concept of the subject and the underlying theme of this text. The concept of risk-neutrality is due in its modern form to Harrison and Pliska (1981) in 1981 – though the idea can be traced back to actuarial practice much earlier (see Esscher (1932) and also Gerber and Shiu (1995)). Harrison and Pliska call $\mathbb{P}^*$ the *reference measure*; Björk (1999) calls it the *risk-adjusted* or *martingale measure*; Dothan (1990) uses *equilibrium price measure*. The term 'risk-neutral' reflects the $\mathbb{P}^*$-martingale property of the risky assets, since martingales model fair games (one can't win systematically by betting on a martingale).

To summarize, we have:

Theorem 4.4.2 (Risk-neutral Pricing Formula). *In an arbitrage-free complete market $\mathcal{M}$, arbitrage prices of contingent claims are their discounted expected values under the risk-neutral (equivalent martingale) measure $\mathbb{P}^*$.*

There exist several variants and ramifications of the results we have presented so far.

Finite, Discrete Time; Finite Probability Space (our model)

Like Harrison and Pliska (1981) in their seminal paper we used several results from functional analysis. Taqqu and Willinger (1987) provide an approach based on probabilistic methods and allowing a geometric interpretation which yields a connection to linear programming. They analyze certain geometric properties of the sample paths of a given vector-valued stochastic process representing the different stock prices through time. They show that under the requirement that no arbitrage opportunities exist, the price increments between two periods can be converted to martingale differences (see Chapter 3) through an equivalent martingale measure. From a probabilistic point of view this provides a converse to the classical notion that 'one cannot win betting on a martingale' by saying 'if one cannot win betting on a process, then it must be a martingale under an equivalent martingale measure'. Furthermore, they give a characterization of complete markets in terms of an extremal property of a probability measure in the convex set $\tilde{\mathcal{P}}(\tilde{S})$ of martingale measures for $\tilde{S}$ (not necessarily equivalent to $\mathbb{P}$):

The market model $\mathcal{M}$ is complete under a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ if and only if $\mathbb{Q}$ is an extreme point of $\tilde{\mathcal{P}}(\tilde{S})$ (i.e. $\mathbb{Q}$ cannot be expressed as a strictly convex combination of two distinct probability measures in $\tilde{\mathcal{P}}(\tilde{S})$).

They also show that the problem of attainability of a simple contingent claim can be viewed and formulated as the 'dual problem' to finding a certain martingale measure for the price process $\tilde{S}$.

Finite, Discrete Time; General Probability Space

The no-arbitrage condition remains equivalent to the existence of an equivalent martingale measure. The first proof of this was given by Dalang, Morton, and Willinger (1990) using deep functional analytic methods (such as measurable selection and measure-decomposition theorems). There exist now several more accessible proofs, in particular by Schachermayer (1992), using more elementary results from functional analysis (orthogonality arguments in properly chosen spaces, see also Kabanov and Kramkov (1995)) and by Rogers (1994), using a method which essentially comes down to maximizing expected utility of gains from trade over all possible trading strategies.

Discrete Time; Infinite Horizon; General Probability Space

Under this setting the equivalence of no-arbitrage opportunities and existence of an equivalent martingale measure breaks down (see Back and Pliska (1991) and Dalang, Morton, and Willinger (1990) for counterexamples). Introducing a weaker regularity concept than no-arbitrage, namely no free lunch with bounded risk – requiring an absolute bound on the maximal loss occurring in certain basic trading strategies (see Schachermayer (1994) for an exact mathematical definition, Kreps (1981) for related concepts) – Schachermayer (1994) established the following beautiful result:

The condition no free lunch with bounded risk is equivalent to the existence of an equivalent martingale measure.

For a recent overview of variants of fundamental asset pricing theorems proved by probabilistic techniques, we refer the reader to Jacod and Shiryaev (1998). We will not pursue these approaches further, but use our finite discrete-time and finite probability space setting to explore several models which are widely used in practice.

Note. We return to these matters in the more complicated setting of continuous time in Chapter 6; see §6.1 and Theorem 6.1.2.

4.5 The Cox-Ross-Rubinstein Model

In this section we consider simple discrete-time financial market models. The development of the risk-neutral pricing formula is particularly clear in this setting since we require only elementary mathematical methods. The link to the fundamental economic principles of the arbitrage pricing method can be obtained equally straightforwardly. Moreover binomial models, by their very construction, give rise to simple and efficient numerical procedures. We start with the paradigm of all binomial models – the celebrated Cox, Ross, and Rubinstein (1979) model (CRR-model).

4.5.1 Model Structure

We take $d = 1$, that is, our model consists of two basic securities. Recall that the essence of the relative pricing theory is to take the price processes of these basic securities as given and price secondary securities in such a way that no arbitrage is possible.

Our time horizon is T and the set of dates in our financial market model is $t = 0, 1, \ldots, T$. Assume that the first of our given basic securities is a (riskless) bond or bank account B, which yields a riskless rate of return $r > 0$ in each time interval $[t, t+1]$, i.e.

$$B(t+1) = (1+r)B(t), \quad B(0) = 1.$$

So its price process is $B(t) = (1+r)^t, \quad t = 0, 1, \ldots, T$. Furthermore, we have a risky asset (stock) S with price process

$$S(t+1) = \begin{cases} (1+u)S(t) & \text{with probability} \quad p, \\ (1+d)S(t) & \text{with probability} \quad 1-p, \end{cases} \qquad t = 0, 1, \ldots, T-1$$

with $-1 < d < u, S_0 \in I\!R_0^+$ (see Figure 4.1 below).

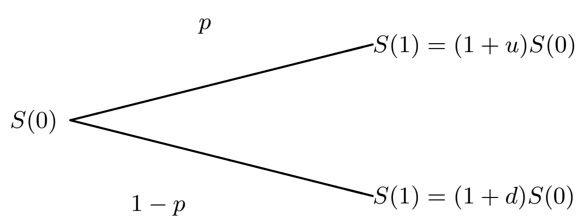

Fig. 4.1. One-step tree diagram

Alternatively we write this as

$$Z(t+1) := \frac{S(t+1)}{S(t)} - 1, \quad t = 0, 1, \ldots, T-1.$$

We set up a probabilistic model by considering the returns process $Z(t),\ t = 1, \ldots, T$ as random variables defined on probability spaces $(\tilde{\Omega}_t, \tilde{\mathcal{F}}_t, \tilde{I\!P}_t)$ with

$$\begin{aligned} \tilde{\Omega}_t &= \tilde{\Omega} = \{d, u\}, \\ \tilde{\mathcal{F}}_t &= \tilde{\mathcal{F}} = \mathcal{P}(\tilde{\Omega}) = \{\emptyset, \{d\}, \{u\}, \tilde{\Omega}\}, \\ \tilde{I\!P}_t &= \tilde{I\!P} \ \text{ with } \ \tilde{I\!P}(\{u\}) = p,\ \tilde{I\!P}(\{d\}) = 1-p,\ p \in (0,1). \end{aligned}$$

On these probability spaces we define

$$Z(t,u) = u \quad \text{and} \quad Z(t,d) = d, \quad t = 1,2,\ldots,T.$$

Our aim, of course, is to define a probability space on which we can model the basic securities (B,S). Since we can write the stock price as

$$S(t) = S(0)\prod_{\tau=1}^{t}(1+Z(\tau)), \quad t = 1,2,\ldots,T,$$

the above definitions suggest using as the underlying probabilistic model of the financial market the *product space* $(\Omega, \mathcal{F}, I\!P)$, see e.g. Williams (1991) Chapter 8, i.e.

$$\Omega = \tilde{\Omega}_1 \times \ldots \times \tilde{\Omega}_T = \tilde{\Omega}^T = \{d,u\}^T,$$

with each $\omega \in \Omega$ representing the successive values of $Z(t),\ t = 1,2,\ldots,T$. Hence each $\omega \in \Omega$ is a T-tuple $\omega = (\tilde{\omega}_1,\ldots,\tilde{\omega}_T)$ and $\tilde{\omega}_t \in \tilde{\Omega} = \{d,u\}$. For the σ-algebra we use $\mathcal{F} = \mathcal{P}(\Omega)$ and the probability measure is given by

$$I\!P(\{\omega\}) = \tilde{I\!P}_1(\{\omega_1\}) \times \ldots \times \tilde{I\!P}_T(\{\omega_T\}) = \tilde{I\!P}(\{\omega_1\}) \times \ldots \times \tilde{I\!P}(\{\omega_T\}).$$

The role of a product space is to model independent replication of a random experiment. The $Z(t)$ above are two-valued random variables, so can be thought of as tosses of a biased coin; we need to build a probability space on which we can model a succession of such independent tosses.

Now we redefine (with a slight abuse of notation) the $Z(t),\ t = 1,\ldots,T$ as random variables on $(\Omega,\mathcal{F},I\!P)$ as (the tth projection)

$$Z(t,\omega) = Z(t,\omega_t).$$

Observe that by this definition (and the above construction) $Z(1),\ldots,Z(T)$ are independent and identically distributed with

$$I\!P(Z(t) = u) = p = 1 - I\!P(Z(t) = d).$$

To model the flow of information in the market we use the obvious filtration

$$\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega\} && \text{(trivial } \sigma\text{-field)},\\
\mathcal{F}_t &= \sigma(Z(1),\ldots,Z(t)) = \sigma(S(1),\ldots,S(t)), &&\\
\mathcal{F}_T &= \mathcal{F} = \mathcal{P}(\Omega) && \text{(class of all subsets of } \Omega).
\end{aligned}$$

This construction emphasizes again that a multi-period model can be viewed as a sequence of single-period models. Indeed, in the Cox-Ross-Rubinstein case we use identical and independent single-period models. As we will see in the sequel this will make the construction of equivalent martingale measures relatively easy. Unfortunately we can hardly defend the assumption of independent and identically distributed price movements at each time period in practical applications.

Remark 4.5.1. We used this example to show explicitly how to construct the underlying probability space. Having done this in full once, we will from now on feel free to take for granted the existence of an appropriate probability space on which all relevant random variables can be defined.

4.5.2 Risk-neutral Pricing

We now turn to the pricing of derivative assets in the Cox-Ross-Rubinstein market model. To do so we first have to discuss whether the Cox-Ross-Rubinstein model is arbitrage-free and complete.

To answer these questions we have, according to our fundamental theorems (Theorems 4.2.1 and 4.3.1), to understand the structure of equivalent martingale measures in the Cox-Ross-Rubinstein model. In trying to do this we use (as is quite natural and customary) the bond price process $B(t)$ as numéraire.

Our first task is to find an equivalent martingale measure $\boldsymbol{Q}$ such that the $Z(1), \ldots, Z(T)$ remain independent and identically distributed, i.e. a probability measure $\boldsymbol{Q}$ defined as a product measure via a measure $\tilde{\boldsymbol{Q}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that $\tilde{\boldsymbol{Q}}(\{u\}) = q$ and $\tilde{\boldsymbol{Q}}(\{d\}) = 1 - q$. We have:

Proposition 4.5.1. *(i) A martingale measure $\boldsymbol{Q}$ for the discounted stock price $\tilde{S}$ exists if and only if*

$$d < r < u. \tag{4.9}$$

(ii) If equation (4.9) holds true, then there is a unique such measure in $\mathcal{P}$ characterized by

$$q = \frac{r-d}{u-d}. \tag{4.10}$$

Proof. Since $S(t) = \tilde{S}(t)B(t) = \tilde{S}(t)(1+r)^t$, we have $Z(t+1) = S(t+1)/S(t) - 1 = (\tilde{S}(t+1)/\tilde{S}(t))(1+r) - 1$. So, the discounted price $(\tilde{S}(t))$ is a $\boldsymbol{Q}$-martingale if and only if for $t = 0, 1, \ldots, T-1$

$$\begin{aligned} & \mathbb{E}^{\boldsymbol{Q}}[\tilde{S}(t+1)|\mathcal{F}_t] = \tilde{S}(t) \Leftrightarrow \mathbb{E}^{\boldsymbol{Q}}[(\tilde{S}(t+1)/\tilde{S}(t))|\mathcal{F}_t] = 1 \\ \Leftrightarrow\ & \mathbb{E}^{\boldsymbol{Q}}[Z(t+1)|\mathcal{F}_t] = r. \end{aligned}$$

But $Z(1), \ldots, Z(T)$ are mutually independent and hence $Z(t+1)$ is independent of $\mathcal{F}_t = \sigma(Z(1), \ldots, Z(t))$. So

$$r = \mathbb{E}^{\boldsymbol{Q}}(Z(t+1)|\mathcal{F}_t) = \mathbb{E}^{\boldsymbol{Q}}(Z(t+1)) = uq + d(1-q)$$

is a weighted average of u and d; this can be r if and only if $r \in [d, u]$. As $\boldsymbol{Q}$ is to be *equivalent* to $\mathbb{P}$ and $\mathbb{P}$ has no non-empty null sets, $r = d, u$ are excluded and (4.9) is proved.

To prove uniqueness and to find the value of q we simply observe that under (4.9)

$$u \times q + d \times (1-q) = r$$

has a unique solution. Solving it for q leads to the above formula. $\square$

From now on we assume that (4.9) holds true. Using the above Proposition we immediately get:

Corollary 4.5.1. *The Cox-Ross-Rubinstein model is arbitrage-free.*

Proof. By Proposition 4.5.1 there exists an equivalent martingale measure and this is by the no-arbitrage theorem (Theorem 4.2.1) enough to guarantee that the Cox-Ross-Rubinstein model is free of arbitrage. $\square$

Uniqueness of the solution of the linear equation (4.7) under (4.9) gives completeness of the model, by the completeness theorem (Theorem 4.3.1):

Proposition 4.5.2. *The Cox-Ross-Rubinstein model is complete.*

One can translate this result – on uniqueness of the equivalent martingale measure – into financial language. Completeness means that all contingent claims can be replicated. If we do this in the large, we can do it in the small by restriction, and conversely, we can build up our full model from its constituent components. To summarize:

Corollary 4.5.2. *The multi-period model is complete if and only if every underlying single-period model is complete.*

We can now use the risk-neutral valuation formula to price *every* contingent claim in the Cox-Ross-Rubinstein model.

Proposition 4.5.3. *The arbitrage price process of a contingent claim X in the Cox-Ross-Rubinstein model is given by*

$$\pi_X(t) = B(t)\mathbb{E}^*\left(X/B(T)|\mathcal{F}_t\right) \quad \forall t = 0, 1, \ldots, T,$$

where $\mathbb{E}^*$ is the expectation operator with respect to the unique equivalent martingale measure $\mathbb{P}^*$ characterized by $p^* = (r-d)/(u-d)$.

Proof. This follows directly from Proposition 4.2.5 since the Cox-Ross-Rubinstein model is arbitrage-free and complete. $\square$

We now give simple formulas for pricing (and hedging) of European contingent claims $X = f(S_T)$ for suitable functions f (in this simple framework all functions $f : \mathbb{R} \to \mathbb{R}$). We use the notation

$$F_\tau(x,p) := \sum_{j=0}^{\tau} \binom{\tau}{j} p^j (1-p)^{\tau-j} f\left(x(1+u)^j(1+d)^{\tau-j}\right) \tag{4.11}$$

Observe that this is just an evaluation of $f(S(j))$ along the probability-weighted paths of the price process. Accordingly, j, $\tau - j$ are the numbers of times $Z(i)$ takes the two possible values d, u.

Corollary 4.5.3. *Consider a European contigent claim with expiry T given by $X = f(S_T)$. The arbitrage price process $\pi_X(t),\ t = 0, 1, \ldots, T$ of the contingent claim is given by (set $\tau = T - t$)*

$$\pi_X(t) = (1 + r)^{-\tau} F_\tau(S_t, p^*). \tag{4.12}$$

Proof. Recall that

$$S(t) = S(0) \prod_{j=1}^{t} (1 + Z(j)), \quad t = 1, 2, \ldots, T.$$

By Proposition 4.5.3 the price $\Pi_X(t)$ of a contingent claim $X = f(S_T)$ at time t is

$$\begin{aligned}
\pi_X(t) &= (1 + r)^{-(T-t)} \mathbb{E}^*[f(S(T))|\mathcal{F}_t] \\
&= (1 + r)^{-(T-t)} \mathbb{E}^* \left[f\left(S(t) \prod_{i=t+1}^{T} (1 + Z(i)) \right) \middle| \mathcal{F}_t \right] \\
&= (1 + r)^{-(T-t)} \mathbb{E}^* \left[f\left(S(t) \prod_{i=t+1}^{T} (1 + Z(i)) \right) \right] \\
&= (1 + r)^{-\tau} F_\tau(S(t), p^*).
\end{aligned}$$

We used the role of independence property of conditional expectations from Proposition 2.5.1 in the next-to-last equality. It is applicable since $S(t)$ is $\mathcal{F}_t$-measurable and $Z(t + 1), \ldots, Z(T)$ are independent of $\mathcal{F}_t$. □

An immediate consequence is the pricing formula for the European call option, i.e. $X = f(S_T)$ with $f(x) = (x - K)^+$.

Corollary 4.5.4. *Consider a European call option with expiry T and strike price K written on (one share of) the stock S. The arbitrage price process $\Pi_C(t),\ t = 0, 1, \ldots, T$ of the option is given by (set $\tau = T - t$)*

$$\Pi_C(t) = (1+r)^{-\tau} \sum_{j=0}^{\tau} \binom{\tau}{j} p^{*j} (1-p^*)^{\tau-j} (S(t)(1+u)^j (1+d)^{\tau-j} - K)^+. \tag{4.13}$$

For a European put option, we can either argue similarly or use put-call parity.

4.5.3 Hedging

Since the Cox-Ross-Rubinstein model is complete we can find unique hedging strategies for replicating contingent claims. Recall that this means we can find a self-financing portfolio $\varphi(t) = (\varphi_0(t), \varphi_1(t))$, φ predictable, such that the value process $V_\varphi(t) = \varphi_0(t)B(t) + \varphi_1(t)S(t)$ satisfies

$$\Pi_X(t) = V_\varphi(t), \quad \text{for all } t = 0, 1, \dots, T.$$

Using the bond as numéraire we get the discounted equation

$$\tilde{\Pi}_X(t) = \tilde{V}_\varphi(t) = \varphi_0(t) + \varphi_1(t)\tilde{S}(t), \quad \text{for all } t = 0, 1, \dots, T.$$

By the pricing formula, Proposition 4.5.3, we know the arbitrage price process and using the restriction of predictability of φ, this leads to a unique replicating portfolio process φ. We can compute this portfolio process at any point in time as follows. The equation $\tilde{\Pi}_X(t) = \varphi_0(t) + \varphi_1(t)\tilde{S}(t)$ has to be true for each $\omega \in \Omega$ and each $t = 1, \dots, T$. Given such a t we only can use information up to (and including) time $t-1$ to ensure that φ is predictable. Therefore we know $S(t-1)$, but we only know that $S(t) = (1 + Z(t))S(t-1)$. However, the fact that $Z(t) \in \{d, u\}$ leads to the following system of equations, which can be solved for $\varphi_0(t)$ and $\varphi_1(t)$ uniquely. Making the dependence of $\tilde{\Pi}_X$ on $\tilde{S}$ explicit, we have

$$\begin{aligned}
\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+u)) &= \varphi_0(t) + \varphi_1(t)\tilde{S}_{t-1}(1+u), \\
\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+d)) &= \varphi_0(t) + \varphi_1(t)\tilde{S}_{t-1}(1+d).
\end{aligned}$$

This gives two simultaneous linear equations in two unknowns, with solution

$$\begin{aligned}
\varphi_0(t) &= \frac{\tilde{S}_{t-1}(1+u)\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+d)) - \tilde{S}_{t-1}(1+d)\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+u))}{\tilde{S}_{t-1}(1+u) - \tilde{S}_{t-1}(1+d)} \\
&= \frac{(1+u)\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+d)) - (1+d)\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+u))}{(u-d)} \\
\varphi_1(t) &= \frac{\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+u)) - \tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+d))}{\tilde{S}_{t-1}(1+u) - \tilde{S}_{t-1}(1+d)} \\
&= \frac{\tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+u)) - \tilde{\Pi}_X(t, \tilde{S}_{t-1}(1+d))}{\tilde{S}_{t-1}(u-d)}.
\end{aligned}$$

Observe that we only need to have information up to time $t-1$ to compute $\varphi(t)$, hence φ is predictable. We make this rather abstract construction more transparent by constructing the hedge portfolio for the European contingent claims.

Proposition 4.5.4. *The perfect hedging strategy $\varphi = (\varphi_0, \varphi_1)$ replicating the European contingent claim $f(S_T)$ with time of expiry T is given by (again using $\tau = T - t$)*

$$\begin{aligned}
\varphi_1(t) &= \frac{(1+r)^{-\tau}\left(F_\tau(S_{t-1}(1+u), p^*) - F_\tau(S_{t-1}(1+d), p^*)\right)}{S_{t-1}(u-d)}, \\
\varphi_0(t) &= \frac{(1+u)F_\tau(S_{t-1}(1+d), p^*) - (1+d)F_\tau(S_{t-1}(1+u), p^*)}{(u-d)(1+r)^T}.
\end{aligned}$$

Proof. $(1+r)^{-\tau}F_\tau(S_t, p^*)$ must be the value of the portfolio at time t if the strategy $\varphi = (\varphi(t))$ replicates the claim:

$$\varphi_0(t)(1+r)^t + \varphi_1(t)S(t) = (1+r)^{-\tau}F_\tau(S_t, p^*).$$

Now $S(t) = S(t-1)(1+Z(t)) = S(t-1)(1+u)$ or $S(t-1)(1+d)$, so:

$$\varphi_0(t)(1+r)^t + \varphi_1(t)S(t-1)(1+u) = (1+r)^{-\tau}F_\tau(S_{t-1}(1+u), p^*),$$

$$\varphi_0(t)(1+r)^t + \varphi_1(t)S(t-1)(1+d) = (1+r)^{-\tau}F_\tau(S_{t-1}(1+d), p^*).$$

Subtract:

$$\varphi_1(t)S(t-1)(u-d) = (1+r)^{-\tau}\left(F_\tau(S_{t-1}(1+u), p^*) - F_\tau(S_{t-1}(1+d), p^*)\right).$$

So $\varphi_1(t)$ in fact depends only on $S(t-1)$, thus yielding the predictability of φ, and

$$\varphi_1(t) = \frac{(1+r)^{-\tau}\left(F_\tau(S_{t-1}(1+u), p^*) - F_\tau(S_{t-1}(1+d), p^*)\right)}{S(t-1)(u-d)}.$$

Using any of the equations in the above system and solving for $\varphi_0(t)$ completes the proof. □

To write the corresponding result for the European call, we use the following notation.

$$C(\tau, x) := \sum_{j=0}^{\tau} \binom{\tau}{j} p^{*j}(1-p^*)^{\tau-j}(x(1+u)^j(1+d)^{\tau-j} - K)^+.$$

Then $(1+r)^{-\tau}C(\tau, x)$ is value of the call at time t (with time to expiry τ) given that $S(t) = x$.

Corollary 4.5.5. *The perfect hedging strategy $\varphi = (\varphi_0, \varphi_1)$ replicating the European call option with time of expiry T and strike price K is given by*

$$\varphi_1(t) = \frac{(1+r)^{-\tau}\left(C(\tau, S_{t-1}(1+u)) - C(\tau, S_{t-1}(1+d))\right)}{S_{t-1}(u-d)},$$

$$\varphi_0(t) = \frac{(1+u)C(\tau, S_{t-1}(1+d)) - (1+d)C(\tau, S_{t-1}(1+u))}{(u-d)(1+r)^T}.$$

Notice that the numerator in the equation for $\varphi_1(t)$ is the difference of two values of $C(\tau, x)$, with the larger value of x in the first term (recall $u > d$). When the payoff function $C(\tau, x)$ is an increasing function of x, as for the European call option considered here, this is non-negative. In this case $\varphi_1(t) \geq 0$: the replicating strategy does not involve short-selling. We record this as:

Corollary 4.5.6. *When the payoff function is a non-decreasing function of the asset price $S(t)$, the perfect-hedging strategy replicating the claim does not involve short-selling of the risky asset.*

If we do not use the pricing formula from Proposition 4.5.3 (i.e. the information on the price process), but only the final values of the option (or more generally of a contingent claim), we are still able to compute the arbitrage price and to construct the hedging portfolio by backward induction. In essence this is again only applying the one-period calculations for each time interval and each state of the world. We outline this procedure for the European call starting with the last period $[T-1,T]$. We have to choose a replicating portfolio $\varphi(T) = (\varphi_0(T), \varphi_1(T))$ based on the information available at time $T-1$ (and so $\mathcal{F}_{T-1}$-measurable). So for each $\omega \in \Omega$ the following equation has to hold:

$$\pi_X(T,\omega) = \varphi_0(T,\omega)B(T,\omega) + \varphi_1(T,\omega)S(T,\omega).$$

Given the information $\mathcal{F}_{T-1}$ we know all but the last coordinate of ω, and this gives rise to two equations (with the same notation as above):

$$\pi_X(T, S_{T-1}(1+u)) = \varphi_0(T)(1+r)^T + \varphi_1(T)S_{T-1}(1+u),$$

$$\pi_X(T, S_{T-1}(1+d)) = \varphi_0(T)(1+r)^T + \varphi_1(T)S_{T-1}(1+d).$$

Since we know the payoff structure of the contingent claim at time T, for example in case of a European call $\pi_X(T, S_{T-1}(1+u)) = ((1+u)S_{T-1} - K)^+$ and $\pi_X(T, S_{T-1}(1+d)) = ((1+d)S_{T-1} - K)^+$, we can solve the above system and obtain

$$\varphi_0(T) = \frac{(1+u)\Pi_X(T, S_{T-1}(1+d)) - (1+d)\Pi_X(T, S_{T-1}(1+u))}{(u-d)(1+r)^T}$$
$$\varphi_1(T) = \frac{\Pi_X(T, S_{T-1}(1+u)) - \Pi_X(T, S_{T-1}(1+d))}{S_{T-1}(u-d)}.$$

Using this portfolio one can compute the arbitrage price of the contingent claim at time $T-1$ given that the current asset price is S_{T-1} as

$$\pi_X(T-1, S_{T-1}) = \varphi_0(T, S_{T-1})(1+r)^{T-1} + \varphi_1(T, S_{T-1})S(T-1).$$

Now the arbitrage prices at time $T-1$ are known and one can repeat the procedure to successively compute the prices at $T-2, \ldots, 1, 0$.

The advantage of our risk-neutral pricing procedure over this approach is that we have a single formula for the price of the contingent claim at all times t at once, and don't have to use a backward induction only to compute a price at a special time t.

4.6 Binomial Approximations

Suppose we observe financial assets during a continuous time period $[0, T]$. To construct a stochastic model of the price processes of these assets (to, e.g. value contingent claims) one basically has two choices: one could model the processes as continuous-time stochastic processes (for which the theory of stochastic calculus is needed) or one could construct a sequence of discrete-time models in which the continuous-time price processes are approximated by discrete-time stochastic processes in a suitable sense. We describe the second approach now by examining the asymptotic properties of a sequence of Cox-Ross-Rubinstein models.

4.6.1 Model Structure

We assume that all random variables subsequently introduced are defined on a suitable probability space $(\Omega, \mathcal{F}, I\!P)$. We want to model two assets, a riskless bond B and a risky stock S, which we now observe in a continuous-time interval $[0, T]$. To transfer the continuous-time framework into a binomial structure we make the following adjustments. Looking at the nth Cox-Ross-Rubinstein model in our sequence, there is a prespecified number k_n of trading dates. We set $\Delta_n = T/k_n$ and divide $[0, T]$ in k_n subintervals of length Δ_n, namely $I_j = [j\Delta_n, (j+1)\Delta_n]$, $j = 0, \ldots, k_n - 1$. We suppose that trading occurs only at the equidistant time points $t_{n,j} = j\Delta_n$, $j = 0, \ldots, k_n - 1$. We fix r_n as the riskless interest rate over each interval I_j, and hence the bond process (in the nth model) is given by

$$B(t_{n,j}) = (1 + r_n)^j, \quad j = 0, \ldots, k_n.$$

In the continuous-time model we compound continuously with spot rate $r \geq 0$ and hence the bond price process $B(t)$ is given by $B(t) = e^{rt}$. In order to approximate this process in the discrete-time framework, we choose r_n such that

$$1 + r_n = e^{r\Delta_n}. \tag{4.14}$$

With this choice we have for any $j = 0, \ldots, k_n$ that $(1 + r_n)^j = \exp(rj\Delta_n) = \exp(rt_{n,j})$. Thus we have approximated the bond process exactly at the time points of the discrete model.

Next we model the one-period returns $S(t_{n,j+1})/S(t_{n,j})$ of the stock by a family of random variables $Z_{n,i}; i = 1, \ldots, k_n$ taking values $\{d_n, u_n\}$ with

$$I\!P(Z_{n,i} = u_n) = p_n = 1 - I\!P(Z_{n,i} = d_n)$$

for some $p_n \in (0, 1)$, which relate to the drift and volatility parameter $\sigma > 0$ of the stock. With these $Z_{n,j}$ we model the stock price process S_n in the nth Cox-Ross-Rubinstein model as

$$S_n(t_{n,j}) = S_n(0) \prod_{i=1}^{j} (1 + Z_{n,i}), \quad j = 0, 1, \ldots, k_n.$$

With the specification of the one-period returns we get a complete description of the discrete dynamics of the stock price process in each Cox-Ross-Rubinstein model. We call such a finite sequence $Z_n = (Z_{n,i})_{i=1}^{k_n}$ a *lattice* or *tree*. The parameters u_n, d_n, p_n, k_n differ from lattice to lattice, but remain constant throughout a specific lattice. In the triangular array $(Z_{n,i}),\ i = 1, \ldots, k_n;\ n = 1, 2, \ldots$ we assume that the random variables are row-wise independent (but we allow dependence between rows). The approximation of a continuous-time setting by a sequence of lattices is called the lattice approach.

It is important to stress that for each n we get a different discrete stock price process $S_n(t)$ and that in general these processes do not coincide on common time points (and are also different from the price process $S(t)$).

Turning back to a specific Cox-Ross-Rubinstein model, we now have as in §4.5 a discrete-time bond and stock price process. We want arbitrage-free financial market models and therefore have to choose the parameters u_n, d_n, p_n accordingly. An arbitrage-free financial market model is guaranteed by the existence of an equivalent martingale measure, and by Proposition 4.5.1 (i) the (necessary and) sufficient condition for that is

$$d_n < r_n < u_n.$$

The risk-neutrality approach implies that the expected (under an equivalent martingale measure) one-period return must equal the one-period return of the riskless bond and hence we get (see Proposition 4.5.1(ii))

$$p_n^* = \frac{r_n - d_n}{u_n - d_n}. \tag{4.15}$$

So the only parameters to choose freely in the model are u_n and d_n. In the next sections we consider some special choices.

4.6.2 The Black-Scholes Option Pricing Formula

We now choose the parameters in the above lattice approach in a special way. Assuming the risk-free rate of interest r as given, we have by (4.14) $1 + r_n = e^{r\Delta_n}$, and the remaining degrees of freedom are resolved by choosing u_n and d_n. We use the following choice:

$$1 + u_n = e^{\sigma\sqrt{\Delta_n}}, \quad \text{and} \quad 1 + d_n = (1 + u_n)^{-1} = e^{-\sigma\sqrt{\Delta_n}}.$$

By Condition (4.15) the risk-neutral probabilities for the corresponding single period models are given by

$$p_n^* = \frac{r_n - d_n}{u_n - d_n} = \frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}}.$$

We can now price contingent claims in each Cox-Ross-Rubinstein model using the expectation operator with respect to the (unique) equivalent martingale measure characterized by the probabilities p_n^* (compare §4.5.2). In particular we can compute the price $\Pi_C(t)$ at time t of a European call on the stock S with strike K and expiry T by Formula (4.13) of Corollary 4.5.4. Let us reformulate this formula slightly. We define

$$a_n = \min\left\{j \in \mathbb{N}_0 | S(0)(1+u_n)^j(1+d_n)^{k_n-j} > K\right\}. \tag{4.16}$$

Then we can rewrite the pricing formula (4.13) for $t = 0$ in the setting of the nth Cox-Ross-Rubinstein model as

$$\begin{aligned}
\Pi_C(0) &= (1+r_n)^{-k_n} \\
&\quad \times \sum_{j=a_n}^{k_n} \binom{k_n}{j} p_n^{*j}(1-p_n^*)^{k_n-j}(S(0)(1+u_n)^j(1+d_n)^{k_n-j} - K) \\
&= S(0)\left[\sum_{j=a_n}^{k_n} \binom{k_n}{j} \left(\frac{p_n^*(1+u_n)}{1+r_n}\right)^j \left(\frac{(1-p_n^*)(1+d_n)}{1+r_n}\right)^{k_n-j}\right] \\
&\quad -(1+r_n)^{-k_n} K \left[\sum_{j=a_n}^{k_n} \binom{k_n}{j} p_n^{*j}(1-p_n^*)^{k_n-j}\right].
\end{aligned}$$

Denoting the binomial cumulative distribution function with parameters (n, p) as $B^{n,p}(.)$ we see that the second bracketed expression is just

$$\bar{B}^{k_n,p_n^*}(a_n) = 1 - B^{k_n,p_n^*}(a_n).$$

Also the first bracketed expression is $\bar{B}^{k_n,\hat{p}_n}(a_n)$ with

$$\hat{p}_n = \frac{p_n^*(1+u_n)}{1+r_n}.$$

That $\hat{p}_n$ is indeed a probability can be shown straightforwardly. Using this notation we have in the nth Cox-Ross-Rubinstein model for the price of a European call at time $t = 0$ the following formula:

$$\Pi_C^{(n)}(0) = S_n(0)\bar{B}^{k_n,\hat{p}_n}(a_n) - K(1+r_n)^{-k_n}\bar{B}^{k_n,p_n^*}(a_n). \tag{4.17}$$

(We stress again that the underlying is $S_n(t)$, dependent on n, but $S_n(0) = S(0)$ for all n.) We now look at the limit of this expression.

Proposition 4.6.1. *We have the following limit relation:*

$$\lim_{n\to\infty} \Pi_C^{(n)}(0) = \Pi_C^{BS}(0)$$

with $\Pi_C^{BS}(0)$ given by the Black-Scholes formula (we use $S = S(0)$ to ease the notation)

$$\Pi_C^{BS}(0) = SN(d_1(S,T)) - Ke^{-rT}N(d_2(S,T)). \tag{4.18}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$d_1(s,t) = \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}},$$

$$d_2(s,t) = d_1(s,t) - \sigma\sqrt{t} = \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

and $N(.)$ is the standard normal cumulative distribution function.

Proof. Since $S_n(0) = S$ (say) all we have to do to prove the proposition is to show

$$\text{(i)} \quad \lim_{n\to\infty} \bar{B}^{k_n,\hat{p}_n}(a_n) = N(d_1(S,T)),$$
$$\text{(ii)} \quad \lim_{n\to\infty} \bar{B}^{k_n,p_n^*}(a_n) = N(d_2(S,T)).$$

These statements involve the convergence of distribution functions.

To show (i) we interpret

$$\bar{B}^{k_n,\hat{p}_n}(a_n) = I\!P\left(a_n \le Y_n \le k_n\right)$$

with (Y_n) a sequence of random variables distributed according to the binomial law with parameters $(k_n, \hat{p}_n)$. We normalize Y_n to

$$\tilde{Y}_n = \frac{Y_n - I\!E(Y_n)}{\sqrt{Var(Y_n)}} = \frac{Y_n - k_n\hat{p}_n}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}} = \frac{\sum_{j=1}^{k_n}(B_{j,n} - \hat{p}_n)}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}},$$

where $B_{j,n},\ j = 1,\ldots,k_n;\ n = 1,2,\ldots$ are row-wise independent Bernoulli random variables with parameter $\hat{p}_n$. Now using the central limit theorem we know that for $\alpha_n \to \alpha,\ \beta_n \to \beta$ we have

$$\lim_{n\to\infty} I\!P(\alpha_n \le \tilde{Y}_n \le \beta_n) = N(\beta) - N(\alpha).$$

By definition we have

$$I\!P\left(a_n \le Y_n \le k_n\right) = I\!P\left(\alpha_n \le \tilde{Y}_n \le \beta_n\right)$$

with

$$\alpha_n = \frac{a_n - k_n\hat{p}_n}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}} \quad \text{and } \beta_n = \frac{k_n(1-\hat{p}_n)}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}}.$$

Using the following limiting relations:

$$\lim_{n\to\infty} \hat{p}_n = \frac{1}{2}, \quad \lim_{n\to\infty} k_n(1-2\hat{p}_n)\sqrt{\Delta_n} = -T\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right),$$

and the defining relation for a_n, Formula (4.16), we get

$$\begin{aligned}
\lim_{n\to\infty} \alpha_n &= \lim_{n\to\infty} \frac{\dfrac{\log(K/S) + k_n\sigma\sqrt{\Delta_n}}{2\sigma\sqrt{\Delta_n}} - k_n\hat{p}_n}{\sqrt{k_n\hat{p}_n(1-\hat{p}_n)}} \\
&= \lim_{n\to\infty} \frac{\log(K/S) + \sigma k_n\sqrt{\Delta_n}(1-2\hat{p}_n)}{2\sigma\sqrt{k_n\Delta_n\hat{p}_n(1-\hat{p}_n)}} \\
&= \frac{\log(K/S) - (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -d_1(S,T).
\end{aligned}$$

Furthermore we have

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \sqrt{k_n\hat{p}_n^{-1}(1-\hat{p}_n)} = +\infty.$$

So $N(\beta_n) \to 1, N(\alpha_n) \to N(-d_1) = 1 - N(d_1)$, completing the proof of (i).

To prove (ii) we can argue in very much the same way and arrive at parameters α_n^* and β_n^* with $\hat{p}_n$ replaced by p_n^*. Using the following limiting relations:

$$\lim_{n\to\infty} p_n^* = \frac{1}{2}, \quad \lim_{n\to\infty} k_n(1-2p_n^*)\sqrt{\Delta_n} = T\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right),$$

we get

$$\begin{aligned}
\lim_{n\to\infty} \alpha_n^* &= \lim_{n\to\infty} \frac{\log(K/S) + \sigma n\sqrt{\Delta_n}(1-2p_n^*)}{2\sigma\sqrt{n\Delta_n p_n^*(1-p_n^*)}} \\
&= \frac{\log(K/S) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -d_2(s,T).
\end{aligned}$$

For the upper limit we get

$$\lim_{n\to\infty} \beta_n^* = \lim_{n\to\infty} \sqrt{k_n(p_n^*)^{-1}(1-p_n^*)} = +\infty,$$

whence (ii) follows similarly. □

By the above proposition we have derived the classical Black-Scholes European call option valuation formula as an asymptotic limit of option prices in a sequence of Cox-Ross-Rubinstein type models with a special choice of parameters. We will therefore call these models discrete Black-Scholes models. Let us mention here that in the continuous-time Black-Scholes model the dynamics of the (stochastic) stock price process $S(t)$ are modeled by a geometric Brownian motion (or exponential Wiener process). The sample paths of this stochastic price process are almost all continuous and the probability law of $S(t)$ at any time t is lognormal. In particular the time T distribution of $\log\{S(T)/S(0)\}$ is $N(T\mu, T\sigma^2)$ (here μ is the growth rate, σ the volatility of the stock). Looking back at the construction of our sequence of Cox-Ross-Rubinstein models we see that

$$\log \frac{S_n(T)}{S(0)} = \sum_{i=1}^{k_n} \log(1 + Z_{n,i}),$$

with $\log(Z_{n,i})$ Bernoulli random variables with

$$\mathbb{P}(\log(1 + Z_{n,i}) = \sigma\sqrt{\Delta_n}) = p_n = 1 - \mathbb{P}(\log(1 + Z_{n,i}) = -\sigma\sqrt{\Delta_n}).$$

By the (triangular array version) of the central limit theorem, we know that $\log \frac{S_n(T)}{S(0)}$ properly normalized converges in distribution to a random variable with standard normal distribution. Doing similar calculations as in the above proposition we can compute the normalizing constants and get

$$\lim_{n\to\infty} \log \frac{S_n(T)}{S(0)} \sim N(T(r - \sigma^2/2), T\sigma^2),$$

i.e. $\frac{S_n(T)}{S(0)}$ is in the limit lognormally distributed.

Using the terminology of weak convergence, we can therefore say that the probability measures $\mathbb{P}^n$ induced by the distributions of $S_n(T)/S(0)$ converge to the probability measure $\mathbb{Q}$ induced by $N(T(r - \sigma^2/2), T\sigma^2)$.

Therefore as a direct consequence of the definition of weak convergence we have

Proposition 4.6.2. *Let X be a contingent claim of the form $X = h(S(T))$ with h a bounded, uniformly continuous real function. Denote by Π_X^n resp. Π_X the time $t = 0$ price of X in the nth discrete-time resp. the continuous-time Black-Scholes market model. Then*

$$\lim_{n\to\infty} \Pi_X^n = \Pi_X.$$

Proof. Writing the pricing formula for the contingent claim using the expectation operator with respect to the risk-neutral probability measures, we have

$$\Pi_X^n = \mathbb{E}_{\mathbb{P}^n}(h(S_n(T))) = \int h d\mathbb{P}^n,$$

resp.

$$\Pi_X = \mathbb{E}_{\mathbb{Q}}(h(S(T))) = \int h d\mathbb{Q}$$

(since the σ-field at $t = 0$ is assumed to be trivial, we can use expectation instead of conditional expectation). The result now follows from the portmanteau theorem of weak-convergence theory (see e.g. the book Billingsley (1968), §1.2). $\square$

Example. Using $h(x) = \max\{0, (K - x)\}$ we get the above convergence for the European put option, and put-call parity gives the result for the European call option (as above). Observe $g(x) = \max\{0, (x - K)\}$ is unbounded, so we can't apply Proposition 4.6.2 to give another direct proof of Proposition 4.6.1.

We now turn briefly to different choices of u_n and d_n and their effects.

4.6.3 Further Limiting Models

As already mentioned, different choices of the sequences (u_n) and (d_n) lead to different asymptotic stock price processes. We briefly discuss two possible choices.

Jump Stock Price Movements

The key to the results in the last section was the weak convergence of the sequence of random variables $\log(\frac{S_n(T)}{S(0)})$. To show this convergence we basically used the De Moivre-Laplace theorem for binomial random variables. We now use another classical limit theorem for binomial random variables - the 'weak law of small numbers' or 'law of rare events', which states that for certain parameters the limiting distribution is a Poisson distribution (compare §2.9). Indeed, if we choose $u_n = u = e^{\zeta}$, $\zeta > 0$ (independent of n) and $d_n = e^{\xi \Delta_n}$ with some $0 < \xi < r$ we have (for large enough n) an arbitrage-free market model with unique risk-neutral probabilities p_n^* given by

$$p_n^* = \frac{\exp(r\Delta_n) - \exp(\xi\Delta_n)}{u - \exp(\xi\Delta_n)} \to 0, \ (n \to \infty).$$

For this lattice approach the step size of an upward move remains constant through all Cox-Ross-Rubinstein models, but the probability it will occur becomes very small. On the other hand, the size of a downward move becomes very small (as $\Delta_n \to 0$, we have $d_n \to 1$), but its probability becomes very close to 1.

Recall that in the sequence of Cox-Ross-Rubinstein models we modeled the stock price at time T as

$$\log \frac{S_n(T)}{S(0)} = \sum_{i=1}^{k_n} \log(Z_{n,i}),$$

with $\log(Z_{n,i})$ Bernoulli random variables. Given the size of the up and down movements and the probabilities p_n^* as above, an application of the law of rare events (see §2.9) shows that the corresponding sequence of equivalent probability measures $\mathbb{P}^n$ of the Cox-Ross-Rubinstein models converges weakly to the probability measure $\mathbb{Q}$ induced by a Poisson distribution with parameter $\lambda = \frac{Tu(r-\xi)}{u-1}$.

We can apply the pormanteau theorem again to find the valuation formula of a European put and use put-call parity to get the pricing formula for a European call. We use the following notation: C_n is the time $t = 0$ price of a European call in the nth Cox-Ross-Rubinstein model with parameters as above and

$$\bar{\Psi}^{\mu}(x) = 1 - \Psi^{\mu}(x-1) = \sum_{i=x}^{\infty} \frac{e^{-\mu}\mu^i}{i!}$$

the complementary Poisson distribution function with parameter μ. With this notation we have the following limiting relation:

$$\lim_{n\to\infty} C_n = S(0)\bar{\Psi}^{\lambda}(x) - Ke^{-rT}\bar{\Psi}^{\frac{\lambda}{u}}(x).$$

The parameter λ is given as above and $x = (\log(K/S(0)) - \xi T)/\log u$.

In the limiting continuous-time model the stock price process has to be modelled in such a way that 'jumps' are possible, i.e. the paths of the stochastic stock price process must allow discontinuities. This is done by using the continuous-time Poisson process (or another point process, see Chapter §5.2). The distribution of the stock price process in the continuous-time model is then log-Poisson. This kind of binomial model was introduced by Cox and Ross (1976); see also Cox and Rubinstein (1985), p. 365 for a somewhat different textbook treatment.

Constant Elasticity of Variance Diffusion

We now allow the up and down movements of the binomial process to differ predictably from period to period. More explicitly we write (using the notation from above)

$$u_n = u_n(S_n(j\Delta_n), \Delta_n) \quad \text{and} \quad d_n = d_n(S_n(j\Delta_n), \Delta_n).$$

To obtain an arbitrage-free market, we have to choose the probabilities in the underlying single-period models according to (4.10), i.e.

$$p^*_{n,j} = p^*_{n,j}(S_n(j\Delta_n)) = \frac{\exp\{r\Delta_n\} - d_n(S_n(j\Delta_n), \Delta_n)}{u_n(S_n(j\Delta_n), \Delta_n) - d_n(S_n(j\Delta_n), \Delta_n)}.$$

This, of course, implies that the equivalent martingale measure for the nth Cox-Ross-Rubinstein model is dependent on the whole family of probabilities $p^*_{n,0}, \ldots, p^*_{n,k_n-1}$.

For instance, if we use the functions

$$u(y,t) = \mu yt + \sigma y^p\sqrt{t} \quad \text{and} \quad d(y,t) = \mu yt - \sigma y^p\sqrt{t}, \quad 0 < p \le 1,$$

and set

$$u_n(S(t),t) = \exp\{u(S(t),t)\} \quad \text{and} \quad d_n(S(t),t) = \exp\{d(S(t),t)\},$$

we have

$$p_{n,j} = \frac{e^{r\Delta_n} - e^{\mu S_n(j\Delta_n)\Delta_n - \sigma S_n^p(j\Delta_n)\sqrt{\Delta_n}}}{e^{\mu S_n(j\Delta_n)\Delta_n + \sigma S_n^p(j\Delta_n)\sqrt{\Delta_n}} - e^{\mu S_n(j\Delta_n)\Delta_n - \sigma S_n^p(j\Delta_n)\sqrt{\Delta_n}}}.$$

With these parameters, one can show that the probability measures $\mathbb{P}^n$ converge weakly to a probability measure $\mathbb{Q}$ induced by a certain gamma-type distribution. This leads to the constant elasticity of variance option pricing formula for the limit of European call option prices at time 0 in the above sequence of Cox-Ross-Rubinstein models:

$$\lim_{n\to\infty} C_{n,0} = S(0)\sum_{i=1}^{\infty} g(i,x)\bar{G}(i+\lambda,y) - Ke^{-rT}\sum_{i=1}^{\infty} g(i+\lambda,x)\bar{G}(i,y).$$

The function $g(i,u)$ is the gamma density function

$$g(i,u) = \frac{e^{-u}u^{i-1}}{(i-1)!},$$

and the function $\bar{G}(i,z)$ the complementary gamma distribution function

$$\bar{G}(i,z) = \int_z^\infty g(i,u)du.$$

The parameters are given as $x = 2\lambda r S(0)^{\frac{1}{\lambda}} e^{rT/\lambda}/(\sigma^2(e^{rT/\lambda}-1))$, $y = 2\lambda r K^{\frac{1}{\lambda}}/(\sigma^2(e^{rT/\lambda}-1))$ and $\lambda = 1/(2(1-p))$.

The corresponding continuous-time stock price dynamics are given by

$$dS(t) = \mu S(t)dt + \sigma S(t)^p dW(t)$$

(where $dW(t)$ denotes the stochastic differential with respect to the Wiener process – we treat this in Chapter 5) and the constant elasticity in the (conditional) variance term (in front of $dW(t)$) gives the name to this model.

Remark 4.6.1. The numerics of the above approximations have been subject to investigation for quite some time (see Broadie and Detemple (1997) and Leisen (1996) for discussion and references). Such numerical schemes are easy to implement, for instance using Mathematica, and the reader is invited to do so.

4.7 American Options

4.7.1 Theory

Consider a general multi-period framework. The holder of an American derivative security can 'exercise' in any period t and receive payment $f(S_t)$

(or more generally a non-negative payment f_t). In order to hedge such an option, we want to construct a self-financing trading strategy φ_t such that for the corresponding value process $V_\varphi(t)$

$$\begin{aligned} &V_\varphi(0) = x \;\; \text{initial capital} \\ &V_\varphi(t) \geq f_t, \quad \forall t. \end{aligned} \tag{4.19}$$

Such a hedging portfolio is minimal, if for a stopping time τ

$$V_\varphi(\tau) = f_\tau.$$

We assume now that we work in a market model $(\Omega, \mathcal{F}, I\!\!F, I\!\!P)$, which is complete with $I\!\!P^*$ the unique martingale measure.

Then for any hedging strategy φ we have that under $I\!\!P^*$

$$M(t) = \tilde{V}_\varphi(t) = \beta(t) V_\varphi(t) \tag{4.20}$$

is a martingale. Thus we can use the STP (Theorem 3.5.1) to find for any stopping time τ

$$V_\varphi(0) = M_0 = I\!\!E^*(\tilde{V}_\varphi(\tau)). \tag{4.21}$$

Since we require $V_\varphi(\tau) \geq f_\tau$ for any stopping time we find for the required initial capital

$$x \geq \sup_{\tau \in \mathcal{T}} I\!\!E^*(\beta(\tau) f_\tau). \tag{4.22}$$

Suppose now that τ^* is such that $V_\varphi(\tau^*) = f_{\tau^*}$; then the strategy φ is minimal, and since $V_\varphi(t) \geq f_t$ for all t we have

$$x = I\!\!E^*(\beta(\tau^*) f_{\tau^*}) = \sup_{\tau \in \mathcal{T}} I\!\!E^*(\beta(\tau) f_\tau). \tag{4.23}$$

Thus (4.23) is a necessary condition for the existence of a minimal strategy φ. We will show that it is also sufficient and call the price in (4.23) the rational price of an American contingent claim.

Now consider the problem of the option writer to construct such a strategy φ. At time T the hedging strategy needs to cover f_T, i.e. $V_\varphi(T) \geq f_T$ is required. At time $T-1$ the option holder can either exercise and receive f_{T-1} or hold the option to expiry, in which case $B(T-1) I\!\!E^*(\beta(T) f_T | F_{T-1})$ needs to be covered. Thus the hedging strategy of the writer has to satisfy

$$V_\varphi(T-1) = \max\{f_{T-1}, B(T-1) I\!\!E^*(\beta(T) f_T | \mathcal{F}_{T-1})\}. \tag{4.24}$$

Using a backward induction argument we can show that

$$V_\varphi(t-1) = \max\{f_{t-1}, B(t-1) I\!\!E^*(\beta(t) V_\varphi(t) | \mathcal{F}_{t-1})\}. \tag{4.25}$$

Considering only discounted values, this leads to

$$\tilde{V}_\varphi(t-1) = \max\{\tilde{f}_{t-1}, I\!\!E^*(\tilde{V}_\varphi(t) | \mathcal{F}_{t-1})\}. \tag{4.26}$$

Thus we see that $\tilde{V}_\varphi(t)$ is the Snell envelope Z_t of $\tilde{f}_t$.

In particular, we know that

$$Z_t = \sup_{\tau \in \mathcal{T}_t} I\!E^*(\tilde{f}_\tau | \mathcal{F}_t) \tag{4.27}$$

and the stopping time $\tau^* = \min\{s \geq t : Z_s = \tilde{f}_s\}$ is optimal. So

$$Z_t = I\!E^*(\tilde{f}_{\tau^*} | \mathcal{F}_t). \tag{4.28}$$

In case $t = 0$ we can use $\tau_0^* = \min\{s \geq 0 : Z_s = \tilde{f}_s\}$, and then

$$x = Z_0 = I\!E^*(\tilde{f}_{\tau_0^*}) = \sup_{\tau \in \mathcal{T}_0} I\!E^*(\tilde{f}_\tau) \tag{4.29}$$

is the rational option price.

We still need to construct the strategy φ. To do this recall that Z is a supermartingale and so the Doob decomposition yields

$$Z = \tilde{M} - \tilde{A} \tag{4.30}$$

with a martingale $\tilde{M}$ and a predictable, increasing process $\tilde{A}$. We write $M_t = \tilde{M}_t B_t$ and $A_t = \tilde{A}_t B_t$. Since the market is complete, we know that there exists a self-financing strategy $\bar{\varphi}$ such that

$$\tilde{M}_t = \tilde{V}_{\bar{\varphi}}(t). \tag{4.31}$$

Also using (4.30) we find $Z_t B_t = V_{\bar{\varphi}}(t) - A_t$. Now on $C = \{(t, \omega) : 0 \leq t < \tau^*(\omega)\}$ we have that Z is a martingale and thus $A_t(\omega) = 0$. Thus we obtain from $\tilde{V}_{\bar{\varphi}}(t) = Z_t$ that

$$\tilde{V}_{\bar{\varphi}}(t) = \sup_{t \leq \tau \leq T} I\!E^*(\tilde{f}_\tau | \mathcal{F}_t) \quad \forall\, (t, \omega) \in C. \tag{4.32}$$

Now τ^* is the smallest exercise time and $\tilde{A}_{\tau^*(\omega)} = 0$. Thus

$$\tilde{V}_{\bar{\varphi}}(\tau^*(\omega), \omega) = Z_{\tau^*(\omega)}(\omega) = \tilde{f}_{\tau^*(\omega)}(\omega). \tag{4.33}$$

Undoing the discounting we find

$$V_{\bar{\varphi}}(\tau^*) = f_{\tau^*} \tag{4.34}$$

and therefore $\bar{\varphi}$ is a minimal hedge.

Now consider the problem of the option holder, how to find the optimal exercise time. We observe that the optimal exercise time must be an optimal stopping time, since for any other stopping time σ (use Proposition 3.6.2)

$$\tilde{V}_\varphi(\sigma) = Z_\sigma > \tilde{f}_\sigma \tag{4.35}$$

and holding the asset longer would generate a larger payoff. Thus the holder needs to wait until $Z_\sigma = \tilde{f}_\sigma$ i.e. (i) of Proposition 3.6.2 is true. On the other hand with ν the largest stopping time (compare Definition 3.6.2) we see that $\sigma \le \nu$. This follows since using $\bar{\varphi}$ after ν with initial capital from exercising will always yield a higher portfolio value than the strategy of exercising later. To see this recall that $V_{\bar{\varphi}} = Z_t B_t + A_t$ with $A_t > 0$ for $t > \nu$. So we must have $\sigma \le \nu$ and since $A_t = 0$ for $t \le \nu$ we see that Z^σ is a martingale. Now criterion (ii) of Proposition 3.6.2 is true and σ is thus optimal. So

Proposition 4.7.1. *A stopping time $\sigma \in \mathcal{T}_t$ is an optimal exercise time for the American option (f_t) if and only if*

$$\mathbb{E}^*(\beta(\sigma) f_\sigma) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}^*(\beta(\tau) f_\tau). \tag{4.36}$$

4.7.2 American Options in the CRR Model

We now consider how to evaluate an American put option in a standard CRR model. We assume that the time interval $[0, T]$ is divided into N equal subintervals of length Δ say. Assuming the risk-free rate of interest r (over [0,T]) as given, we have $1 + \rho = e^{r\Delta}$ (where we denote the risk-free rate of interest in each subinterval by ρ). The remaining degrees of freedom are resolved by choosing u and d as follows:

$$1 + u = e^{\sigma\sqrt{\Delta}}, \quad \text{and} \quad 1 + d = (1 + u)^{-1} = e^{-\sigma\sqrt{\Delta}}.$$

By condition (4.10), the risk-neutral probabilities for the corresponding single period models are given by

$$p^* = \frac{\rho - d}{u - d} = \frac{e^{r\Delta} - e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}} - e^{-\sigma\sqrt{\Delta}}}.$$

Thus the stock with initial value $S = S(0)$ is worth $S(1+u)^i(1+d)^j$ after i steps up and j steps down. Consequently, after N steps, there are $N + 1$ possible prices, $S(1+u)^i(1+d)^{N-i}$ $(i = 0, \ldots, N)$. There are 2^N possible paths through the tree. It is common to take N of the order of 30, for two reasons:

- typical lengths of time to expiry of options are measured in months (9 months, say); this gives a time step around the corresponding number of days,
- 2^{30} paths is about the order of magnitude that can be comfortably handled by computers (recall that $2^{10} = 1,024$, so 2^{30} is somewhat over a billion).

We can now calculate both the value of an American put option and the optimal exercise strategy by working backwards through the tree (this method of backward recursion in time is a form of the dynamic programming

(DP) technique, due to Richard Bellman, which is important in many areas of optimization and Operational Research).

1. Draw a binary tree showing the initial stock value and having the right number, N, of time intervals.
2. Fill in the stock prices: after one time interval, these are $S(1+u)$ (upper) and $S(1+d)$ (lower); after two time intervals, $S(1+u)^2$, S and $S(1+d)^2 = S/(1+u)^2$; after i time intervals, these are $S(1+u)^j(1+d)^{i-j} = S(1+u)^{2j-i}$ at the node with j 'up' steps and $i-j$ 'down' steps (the '(i,j)' node).
3. Using the strike price K and the prices at the terminal nodes, fill in the payoffs $f^A_{N,j} = \max\{K - S(1+u)^j(1+d)^{N-j}, 0\}$ from the option at the terminal nodes underneath the terminal prices.
4. Work back down the tree, from right to left. The no-exercise values f_{ij} of the option at the (i,j) node are given in terms of those of its upper and lower right neighbours in the usual way, as discounted expected values under the risk-neutral measure:

$$f_{ij} = e^{-r\Delta}[p^* f^A_{i+1,j+1} + (1-p^*) f^A_{i+1,j}].$$

The intrinsic (or early-exercise) value of the American put at the (i,j) node – the value there if it is exercised early – is

$$K - S(1+u)^j(1+d)^{i-j}$$

(when this is nonnegative, and so has any value). The value of the American put is the higher of these:

$$\begin{aligned} f^A_{ij} &= \max\{f_{ij}, K - S(1+u)^j(1+d)^{i-j}\} \\ &= \max\left\{e^{-r\Delta}(p^* f^A_{i+1,j+1} + (1-p^*) f^A_{i+1,j}), K - S(1+u)^j(1+d)^{i-j}\right\}. \end{aligned}$$

5. The initial value of the option is the value f^A_0 filled in at the root of the tree.
6. At each node, it is optimal to exercise early if the early-exercise value there exceeds the value f_{ij} there of expected discounted future payoff.

Note. The above procedure is simple to describe and understand, and simple to program. It is laborious to implement numerically by hand, on examples big enough to be non-trivial. Numerical examples are worked through in detail in Hull (1999), p.359–360 and Cox and Rubinstein (1985), p.241–242.

Mathematically, the task remains of describing the *continuation region* – the part of the tree where early exercise is not optimal. This is a classical *optimal stopping problem*, and as we mentioned above, a solution by explicit formulas is not known – indeed, is probably not feasible. It would take us too far afield to pursue such questions here; for a fairly thorough (but quite difficult) treatment, see Shiryaev et al. (1995). We return to the theory of American options in the continuous-time context in §6.3.1.

We conclude by showing the equivalence of American and European calls without using arbitrage arguments.

Theorem 4.7.1. *Let $(Z_n)_0^N$ be the payoff sequence of an American option. Then $h = Z_N$ is the payoff of the corresponding European option. Write $C_A(n), C_E(n)$ for the values at time n of the American and European options. Then*

(i) $C_A(n) \geq C_E(n)$,
(ii) If $C_E(n) \geq Z_n$, then $C_A(n) = C_E(n)$.

Proof. (i) We use the supermartingale resp. martingale property of the price processes of the discounted American resp. European call to get

$$\tilde{C}_A(n) \geq I\!E^* \left(\tilde{C}_A(N)|\mathcal{F}_n\right) = I\!E^* \left(\tilde{C}_E(N)|\mathcal{F}_n\right) = \tilde{C}_E(n).$$

(ii) $(\tilde{C}_E(n))$ is a P^*-martingale, so in particular a P^*-supermartingale. Being the Snell envelope of (Z_n), $(\tilde{C}_A(n))$ is the least P^*-supermartingale dominating (Z_n). So if $\tilde{C}_E(n) \geq Z_n$ as in the condition of the theorem, $\tilde{C}_E(n) \geq \tilde{C}_A(n)$, so $\tilde{C}_E(n) = \tilde{C}_A(n)$. □

Corollary 4.7.1. *In the Black-Scholes model with one risky asset, the American call option is equivalent to its European counterpart.*

Proof. Here $Z_n = (S_n - K)^+$. Discounting,

$$\begin{aligned}\tilde{C}_E(n) &= (1+\rho)^{-N} I\!E^* \left((S_N - K)^+|\mathcal{F}_n\right) \\ &\geq I\!E^* \left(\tilde{S}_N - K(1+\rho)^{-N}|\mathcal{F}_n\right) = \tilde{S}_n - K(1+\rho)^{-N},\end{aligned}$$

as $\tilde{S}_n$ is a $I\!P^*$-martingale. Without the discounting, this says

$$C_E(n) \geq S_n - K(1+\rho)^{-(N-n)}.$$

This gives $C_E(n) \geq S_n - K$; also $C_E(n) \geq 0$; so $C_E(n) \geq (S_n - K)^+ = Z_n$, and the result follows from the theorem. □

4.8 Further Contingent Claim Valuation in Discrete Time

4.8.1 Barrier Options

Barrier options are options whose payoff depends on whether or not the stock price attains some specified level before expiry. We will be brief here, referring to §6.3.3 for a more extensive discussion of barrier options in continuous time. The simplest case is that of a single, constant barrier at level H. The option may pay ('knock in') or not ('knock out') according as to whether or not level H is attained, from below ('up') or above ('down'). There are thus four possibilities – 'up and in', 'up and out', 'down and in', 'down and out' – for the basic – single, constant barrier – case. In addition, one may have two barriers, with the option knocking in (or out) if the price reaches *either* a lower barrier H_1 *or* an upper barrier H_2. More generally, one may have non-constant – 'moving' – barriers, with the level a function of time.

As always, it pays to be flexible, and to be able to work in discrete or continuous time, as seems more appropriate for the problem in hand. For a full treatment in continuous time, see Zhang (1997), Chapters 10, 11, or §6.3.3. Now a continuous-time price process model, such as the Black-Scholes model based on geometric Brownian motion (§6.2), may be approximated in various ways by discrete-time models (such as the discrete Black-Scholes model, the Cox-Ross-Rubinstein binomial tree model of §4.5); for the passage from discrete to continuous time, see §4.6 (and more generally, §5.9 below).

When we have a barrier option in discrete time, we price it as with the American options of §6.3.1 by backward induction. Some sample paths hit the barriers, and for these we can fill in the payoff from the boundary conditions that define the barriers; as before, we fill in the payoff at the terminal nodes at expiry. We then proceed backwards in time recursively, at each stage using all current information to fill in, as before, the payoffs at new nodes one time step earlier. When we reach the root, the payoff is the value of the option initially.

Problems may easily be encountered when dealing with barrier options in discrete time if the discretization process is not chosen and handled with care. A new discretization process, due to Rogers and Stapleton (1998), proceeds by first discretizing *space*, by steps $\delta x > 0$, and then discretizing *time*, into $\tau_0, \tau_1, \cdots$, where

$$\tau_0 := 0, \quad \tau_{n+1} := \inf\{t > \tau_n : |X(t) - X(\tau_n)| > \delta x\}, \quad n \geq 0,$$

and deal with the resulting random walk (ξ_n), where

$$\xi_n := X(\tau_n).$$

This approximation scheme is accurate, reasonably fast, and very flexible: it is capable of handling a wide variety of problems, with moving as well as fixed barriers. For the theory, and detailed comparison with other available methods, see Rogers and Stapleton (1998); another approach is due to Ait-Sahlia and Lai (1998b). Techniques useful here include continuity corrections for approximations to normality, Edgeworth expansions, and Richardson extrapolation.

4.8.2 Lookback Options

Lookback – or hindsight – options, which we discuss in more detail in §6.3.4 in continuous time, are options that convey the right to 'buy at the low, sell at the high' – in other words, to eliminate the regret that an investor operating in real time on current, partial knowledge would feel looking back in time with complete knowledge. Again, most of the theory is for continuous time (see e.g. Zhang (1997), Chapter 12), but a discrete-time framework may be preferred – or needed, if the only prices available are those sampled at certain discrete time-points. Care is obviously needed here, as discretization of time will miss the extremes of the peaks and troughs giving the highs and lows in continuous time.

Discrete lookback options have been studied from several viewpoints; see e.g. Heynen and Kat (1995), Kat (1995) and Levy and Mantion (1997). An interesting approach using duality theory for random walks has been given by AitSahlia and Lai (1998a).

4.8.3 A Three-period Example

Assume we have two basic securities: a risk-free bond and a risky stock. The one-year risk-free interest rate (continuously compounded) is $r = 0.06$ and the volatility of the stock is 20%. We price calls and puts in a three-period Cox-Ross-Rubinstein model. The up and down movements of the stock price are given by

$$1 + u = e^{\sigma\sqrt{\Delta}} = 1.1224 \quad \text{and} \quad 1 + d = (1+u)^{-1} = e^{-\sigma\sqrt{\Delta}} = 0.8910,$$

with $\sigma = 0.2$ and $\Delta = 1/3$. We obtain risk-neutral probabilities by (4.10)

$$p^* = \frac{e^{r\Delta} - d}{u - d} = 0.5584.$$

We assume that the price of the stock at time $t = 0$ is $S(0) = 100$. To price a European call option with maturity one year ($N = 3$) and strike $K = 10$) we can either use the valuation formula (4.13) or work our way backwards through the tree. Prices of the stock and the call are given in Figure 4.2 below. One can implement the simple evaluation formulae for the CRR- and the BS-models and compare the values. Figure 4.3 is for $S = 100, K = 90, r = 0.06, \sigma = 0.2, T = 1$.

To price a European put, with price process denoted by $p(t)$, and an American put, $P(t)$, (maturity $N = 3$, strike 100), we can for the European put either use the put-call parity (1.1), the risk-neutral pricing formula, or work backwards through the tree. For the prices of the American put we use the technique outlined in §4.8.1. Prices of the two puts are given in Figure 4.4. We indicate the early exercise times of the American put in bold type. Recall that the discrete-time rule is to exercise if the intrinsic value $K - S(t)$ is larger than the value of the corresponding European put.

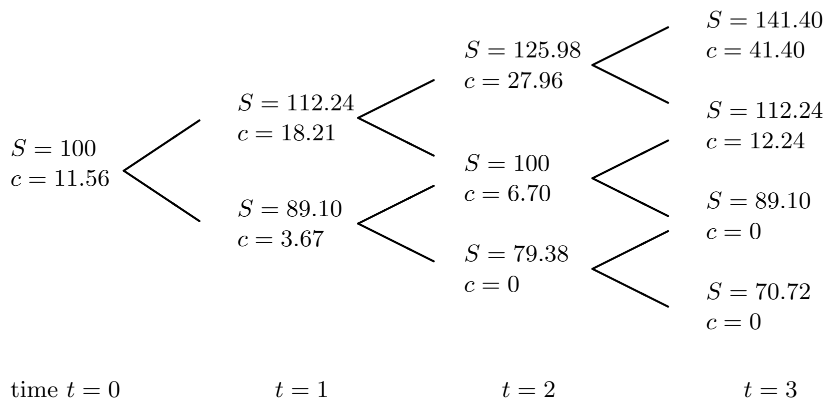

Fig. 4.2. Stock and European call prices

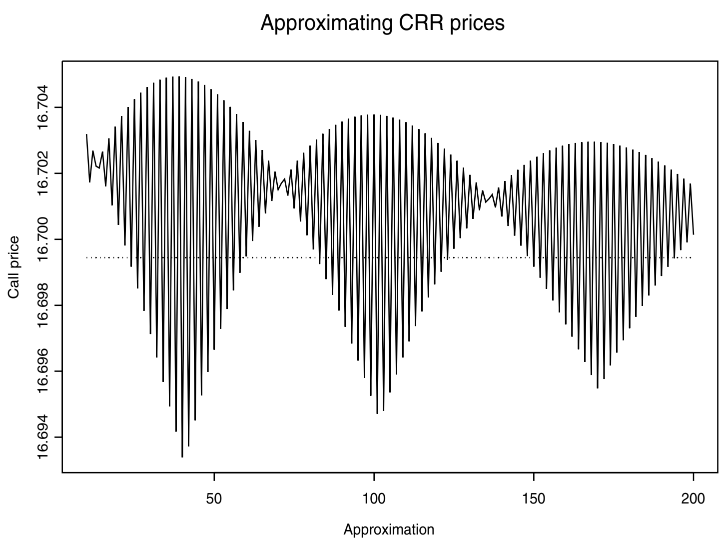

Fig. 4.3. Approximation of Black-Scholes price by Binomial models

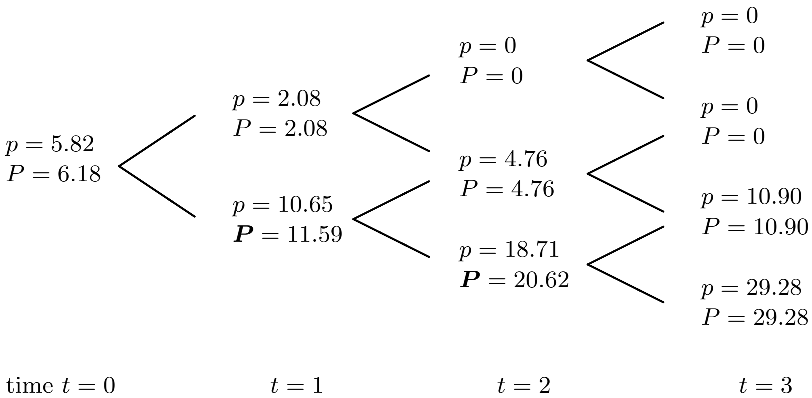

Fig. 4.4. European $p(.)$ and American $P(.)$ put prices

4.9 Multifactor Models

We now discuss examples of discrete-time financial market models with more than two underlying assets. Such models are useful for the evaluation of multivariate contingent claims, such as options on multiple assets (options on the maximum of two or more asset prices, dual-strike options, and portfolio or basket options). For the exposition we assume $d + 1$ financial assets $S_0, S_1, \dots, S_d$. We assume $S_0 = B$, a risk-free bank account or bond, and use B as numéraire.

4.9.1 Extended Binomial Model

This model, proposed by Boyle, Evnine, and Gibbs (1989), uses a single binomial tree for each of the underlying d risky assets. So we have 2^d branches per node. We discuss the case $d = 2$ (i.e. the model consists of two risky assets and the bank account) in detail; the generalization to $d > 2$ is straightforward. To show that this model is arbitrage-free we have to find an equivalent martingale measure, and to show that it is complete we have to prove uniqueness of the equivalent martingale measure. A similar argument to that for the Cox-Ross-Rubinstein model shows that the multi-period extended binomial model is arbitrage-free (complete) if and only if the single-period model is (compare §4.5.2). So it is enough to discuss the single-period model with trading dates $t = 0$ and $t = 1 (= T)$. We assume a risk-free rate of return of $r \geq 0$, so $B(0) = 1$ and $B(1) = 1 + r$. Furthermore we have two risky assets, S_1 and S_2. Since both risky assets are modeled by single binomial trees, we have four possible states of the world at time $t = 1$ with values of $(S_1(1), S_2(1))$ given by $(u_1S_1(0), u_2S_2(0))$ with probability p_{uu}, $(u_1S_1(0), d_2S_2(0))$ with probability p_{ud}, $(d_1S_1(0), u_2S_2(0))$ with probability p_{du} and $(d_1S_1(0), d_2S_2(0))$ with probability p_{dd}, where we assume $u_i > d_i, i = 1, 2$ and positive probabilities. Under the risk-neutral probabilities $p^*_{uu}, p^*_{ud}, p^*_{du}, p^*_{dd}$ the discounted stock price processes $\tilde{S}_i(t) = S_i(t)/B(t)$ have to be martingales. These martingale conditions imply the following two equations:

$$\begin{aligned}
\mathbb{E}[\tilde{S}_1(1)] = \tilde{S}_1(0) &\Leftrightarrow (p^*_{uu} + p^*_{ud})u_1 + (p^*_{du} + p^*_{dd})d_1 = (1+r),\\
\mathbb{E}[\tilde{S}_2(1)] = \tilde{S}_2(0) &\Leftrightarrow (p^*_{uu} + p^*_{du})u_2 + (p^*_{ud} + p^*_{dd})d_2 = (1+r).
\end{aligned}$$

Furthermore, besides the fact that the p^* have to be positive to generate an equivalent measure, we must have

$$p^*_{uu} + p^*_{ud} + p^*_{du} + p^*_{dd} = 1.$$

So we have three equations for the unknown probabilities $p^*_{uu}, p^*_{ud}, p^*_{du}, p^*_{dd}$ and in general (depending on the parameters u_1, d_1, u_2, d_2, r) we will have several (even infinitely many) solutions of the system of the equations above. This means that the extended binomial model is arbitrage-free, but not complete (in accordance to our rule of thumb (§1.4) that we should have as many financial assets to trade in as states of the world).

4.9.2 Multinomial Models

The extended binomial model shows that while it is tempting to model each asset by a single binomial tree, we lose the desirable property of market completeness in doing so. We will therefore now construct an arbitrage-free, complete market model (with $d > 2$ financial assets) following the informal rule of allowing as many different states of the world as we have assets to trade in. Furthermore the stochastic stock price processes in this model can be constructed to be of Markovian nature, that is, rather than the single-period returns being independent unconditionally, they are independent given the present value of the process. This also allows for a more realistic representation of the true prices and is more in line with the most prominent continuous-time model, the Black-Scholes market model, in which the stock price processes are Markovian. We follow an approach that is basically due to He (1990). Again we only discuss the $d = 2$ case (with the risk-free bank account B, with rate of return $r \geq 0$, as numéraire asset and two risky assets S_1, S_2); the case $d > 2$ follows by the same prescription. Let us start with the single-period model. As in the extended binomial case above we assume trading dates $t = 0$ and $t = 1 (= T)$, but now we have only three possible states of the world at time $t = 1$. Indeed we set

$$S_1(1) = S_1(0)Z_1 \quad \text{and} \quad S_2(1) = S_2(0)Z_2,$$

with

$$\mathbb{P}(Z_1 = u_{11}, Z_2 = u_{21}) = p_1; \quad \mathbb{P}(Z_1 = u_{12}, Z_2 = u_{22}) = p_2;$$
$$\mathbb{P}(Z_1 = u_{13}, Z_2 = u_{23}) = p_3.$$

In general Z_1 and Z_2 are not independent, but we still can choose u_{ij} in such a way that they are uncorrelated. Under the risk-neutral probabilities p_1^*, p_2^*, p_3^*, the discounted stock price processes $\tilde{S}_i(t) = S_i(t)/B(t)$ have to be martingales. These martingale conditions imply the following two equations:

$$\mathbb{E}[\tilde{S}_1(1)] = \tilde{S}_1(0) \Leftrightarrow u_{11}p_1^* + u_{12}p_2^* + u_{13}p_3^* = (1+r),$$
$$\mathbb{E}[\tilde{S}_2(1)] = \tilde{S}_2(0) \Leftrightarrow u_{21}p_1^* + u_{22}p_2^* + u_{23}p_3^* = (1+r).$$

Furthermore, besides the fact that the p^* have to be positive to generate an equivalent measure, we must have

$$p_1^* + p_2^* + p_3^* = 1.$$

Therefore we have three equations for the three unknown probabilities and in general (given reasonable parameters u_{ij}) we will have a unique solution of the system of the equations above, and hence an arbitrage-free, complete financial market model.

In the multi-period setting with time horizon T and the set of trading dates given by $\{0 = t_0 < t_1 < \ldots < t_n = T\}$ of equidistant time points with distance Δ_n (observe that we have n time steps), we model the stock price processes by

$$S_i(t_k) = S_i(0) \prod_{j=1}^{k} Z_{ij}, \; k = 0, 1, \ldots, n, \; i = 1, 2,$$

with a sequence of independent random vectors $(Z^{(j)})_{1 \le j \le n}$ such that $Z_1^{(j)}$, $Z_2^{(j)}$ are uncorrelated (but possibly dependent) and

$$\mathbb{P}(Z_1^{(j)} = u_{11}^{(j)}, Z_2^{(j)} = u_{21}^{(j)}) = p_1^{(j)};$$
$$\mathbb{P}(Z_1^{(j)} = u_{12}^{(j)}, Z_2^{(j)} = u_{22}^{(j)}) = p_2^{(j)};$$
$$\mathbb{P}(Z_1^{(j)} = u_{13}^{(j)}, Z_2^{(j)} = u_{23}^{(j)}) = p_3^{(j)}.$$

Since for each j the random vector $Z^{(j)}$ can be in one of three possible states, the above argument applies for each ‘underlying’ single-period market and the multi-period market is arbitrage-free and complete.

The most important case here is $Z_i^{(j+1)} = u_i(S(t_j), t_j, \epsilon^{(j)}), \; i = 1, 2, \; j = 0, \ldots n-1$, with a sequence of independent random vectors $(\epsilon^{(j)})_{j \le, n-1}$ such that $\epsilon_1^{(j)}, \epsilon_2^{(j)}$ are uncorrelated (but possibly dependent) and sufficiently smooth functions u_i. Then $u_i^{(j+1)}$ are predictable functions of $S(t_j)$ making the discrete stochastic process $S_i(t)$ Markovian. We will construct a financial market model of this type in §6.4.

Exercises

4.1 Construct hedging strategies for the European call and put in the setting of the example in §4.8.4.

4.2 Compare the Black-Scholes price with Cox-Ross-Rubinstein price approximations. Is the convergence of Cox-Ross-Rubinstein prices to the Black-Scholes price 'smooth' or 'oscillating'? (See (Leisen 1996) for details.)

4.3 Consider a European call option, written on a stock S with strike price 100, that matures in one year. Assume the continuously compounded risk-free interest rate is 5%, the current price of the stock is 90 and its volatility is $\sigma = 0.2$.

1. Set up a three-period binomial (Cox-Ross-Rubinstein) model for the stock price movements.
2. Compute the risk-neutral probabilities and find the value of the call at each node.
3. Construct a hedging portfolio for the call.

4.4 Consider put options, written on a stock S, with strike price 100 that mature in one year. Assume the continuously compounded risk-free interest rate is 6%, the current price of the stock is 100 and its volatility is $\sigma = 0.25$.

1. Set up a three-period binomial (Cox-Ross-Rubinstein) model for the stock price movements.
2. Compute the risk-neutral probabilities and find the value of a European put at each node.
3. Construct a hedging portfolio for the European put.
4. Now compute the values of a corresponding American put at each node and set up a hedging portfolio. Compare with the hedging portfolio in 3.

4.5 Consider a European powered call option, written on a stock S, with expiry T and strike K. The payoff is ($p > 1$):

$$C_p(T) = \begin{cases} (S(T) - K)^p, & S(T) \geq K; \\ 0 & S(T) < K. \end{cases}$$

Assume that $T = 1$ year, $S(0) = 90, \sigma = 0.3, K = 100$. Consider a two-period binomial model.

1. Price C_p using the risk-neutral valuation formula.
2. Construct a hedge portfolio and compute arbitrage prices (which of course will agree with the risk-neutral prices) using the hedging portfolio.
3. Compare the hedge portfolio with a hedge portfolio for a usual European call. What are the implications for the risk-management of powered call options?

4.6 In *static hedging* of exotic options, one tries to construct a portfolio of standard options – with varying strikes and maturities but fixed weights that will not require any further adjustment – that will exactly replicate the value of the given target option for a chosen range of future times and market levels.

We will construct a static hedge for a barrier option in a binomial five-period model. Consider a zero interest-rate world with a stock worth 100 today. The stock price can move up and down 10 with probability 0.5 at the end of a fixed period.

Our target for replication is a five-period up-and-out European-style call with a strike of 70 and a barrier of 120. This option has natural boundaries both at expiration in five periods and on the knockout barrier at 120.

Create a portfolio of ordinary options that collectively have the same payoff as the up-and-out call on the boundaries. To create such a portfolio follow the steps:

1. Start with an ordinary call struck at 70. It has the same payoff if the barrier is never reached.
2. Add a short position in 10 five-period calls with strike 120 to the portfolio to make the portfolio value 0 at the time 4 boundary point.
3. Add a long position in 5 three-period calls struck at 120 to complete the portfolio.

For each portfolio, compute the value-process at every node and compare it with the value of the barrier option.