Math 236A, Fall 2022.

Final Exam

Due: Wed., Dec. 7, 2022, 4pm. Please leave the assignment in my mailbox, slide it under the door of my office, or send me a single pdf file by email. *Make a credible attempt at every problem.* You can only use your notes for this course when solving the problems.

0. Report your homework score: award yourself a point for every homework problem which you essentially solved, by your own judgment.

1. Let $X_t = \int_0^t e^{|B_s|}\, dB_s$. Show that the stochastic integral exists for all $t \geq 0$, and that X_t is a martingale. Also, show that $\mathrm{Var}(X_t) \geq t$ for all t. Finally, show $P(X_t = 0 \text{ for infinitely many } t) = 1$.

2. Consider the linear stochastic differential equation for exponentially damped multiplicative noise: $dX_t = e^{-ct} X_t\, dB_t$, $X_0 = 1$, for a constant $c > 0$. Find the solution for this equation (which may involve Riemann integrals), and show that it converges almost surely as $t \to \infty$ to a random limit X_∞. Identify the distribution of X_∞ as a function of a standard Normal random variable.

3. Consider the two processes

$$X_t = \int_0^t \cos(s)\, dB_s^1 \qquad \text{and} \qquad Y_t = \int_0^t \sin(s)\, dB_s^2.$$

(a) Assume first that $B_t^1 = B_t^2 = B_t$ is the same Brownian motion. For which times t are the random variables X_t and Y_t independent? (*Hint*. Determine the type of distribution (X_t, Y_t) has and use Itô product formula.)
(b) Assume now that B_t^1 and B_t^2 are independent Brownian motions. Compute the bracket $\langle Z \rangle_t$ of the process $Z_t = X_t + Y_t$.

4. (a) Show that for any $\epsilon > 0$, $P(|B_t| < \epsilon \text{ for all } t \in [0, 1]) > 0$.
(b) Let $h : [0, 1] \to \mathbb{R}$ be a continuously differentiable function with $h(0) = 0$. Show that for any $\epsilon > 0$, $P(|B_t - h(t)| < \epsilon \text{ for all } t \in [0, 1]) > 0$. (*Hint*: Girsanov. Use (a) even if you did not prove it.)
(c) Let h be as in (b). Show that, with probability 1, there exists a (random) time T so that $|B_{T+t} - h| < \epsilon$ for all $t \in [0, 1]$. ("The Brownian motion will eventually draw the graph of h.")

5. Consider logistic growth with multiplicative noise, that is, the SDE

$$dX_t = (\lambda X_t - X_t^2)\, dt + \sigma X_t\, dB_t.$$

Here, $\lambda > 0$ and $\sigma > 0$ are fixed constants, and $X_0 = x_0 > 0$ is deterministic. Show that this equation has a unique solution satisfying $X_t > 0$, by deriving the equation for $Y_t = 1/X_t$ and solving it explicitly. What can you say about the behavior of X_t as $t \to \infty$?