Math 236A, Fall 2022.

Homework 1

Due: Oct. 5, 2022

1. Assume that P and Q are equivalent probability measures on a space $(\Omega, \mathcal{F})$, and let $\Lambda = \frac{dQ}{dP}$. Let X be a random variable with $E_P|X| < \infty$ and $E_Q|X| < \infty$. Prove a Bayes' formula: for a sigma algebra $\mathcal{G} \subset \mathcal{F}$,

$$E_Q(X \mid \mathcal{G}) = \frac{E_P(X\Lambda \mid \mathcal{G})}{E_P(\Lambda \mid \mathcal{G})},$$

Q-a.s. (or equivalenlty P-a.s.). (First show that all conditional expectations exists and that the denominator is a.s. nonzero.)

2. Assume a discrete financial market with $T = 1$, $d = 2$, $S_0^0 = S_1^0 = 1$, $(S_0^1, S_0^2) = (2/3, 2)$, and that the vector (S_1^1, S_1^2) attains only the following three values, each with positive probability: $(0, a)$, $(1, b)$, $(1, 3)$. For which values of parameters $a, b \geq 0$ is the market viable? Complete? When the market is viable and complete, describe the replication of an arbitrary contingent claim.

3. Assume that a discrete financial market is given by $S_t = (S_t^0, \ldots, S_t^d)$, $t = 0, \ldots, T$, and that the numeraire S_t^0 is deterministic. (Recall that we are assuming $\mathcal{F}_0 = \{0, \Omega\}$.)
(a) Assume the market is viable. Let h be an attainable contingent claim. Assume also that h is $\mathcal{F}_t$-measurable for some $t < T$. Show that the h is also attainable as a contingent claim on the interval $[0, t]$ (i.e., it can be replicated not only at time T, but also at time t).
(b) Assume that the market is viable and complete. Show that the filtration must be generated by the (vector of) prices alone, i.e., $\mathcal{F}_t = \sigma(S_0, \ldots, S_t)$.

4. Assume that a discrete financial market, given by $S_t = (S_t^0, \ldots, S_t^d)$, $t = 0, \ldots, T$, is viable and complete. As in class, let $\widetilde{S}_t = \frac{1}{S_t^0} S_t$. Assume that also $S_t^1 > 0$ for all t, and let $\overline{S}_t = \frac{1}{S_t^1} S_t$. Let $\widetilde{Q}$ be the unique measure that makes $\widetilde{S}_t$ a martingale. Denote

$$\Lambda_t = \frac{S_t^1 S_0^0}{S_t^0 S_0^1}.$$

(a) Show that Λ_t is a $\widetilde{Q}$-martingale.
(b) Show that the unique measure $\overline{Q}$ that makes $\overline{S}_t$ a martingale is given by

$$\frac{d\overline{Q}}{d\widetilde{Q}} = \Lambda_T.$$

(c) Show also that for any contingent claim h,

$$E_{\widetilde{Q}}\left(h \cdot \frac{S_0^0}{S_T^0}\right) = E_{\overline{Q}}\left(h \cdot \frac{S_0^1}{S_T^1}\right).$$

This is known as the *Principle of numeraire equivalence*.

5. Assume a viable *binary* market. That is, the market has: $d = 1$; deterministic $\beta_t = 1/S_t^0$; one stock $S_t = S_t^1$, in which the conditional distribution of S_{t+1} given S_t is concentrated on two values; and natural filtration. Viability implies that, given $\widetilde{S}_t = a$, the two values of $\widetilde{S}_{t+1}$ are on different sides of a. Recall that such market is automatically complete. Let Q be the risk-neutral measure under which $\widetilde{S}_t$ is a martingale. Assume that M_t is a Q-martingale. Prove that M_t is a martingale transform of $\widetilde{S}_t$.