Math 236A, Fall 2022.

Homework Assignment 2

Due: Oct. 12, 2022

1. Below are the scanned pages 46 and 47 from the book "Financial Calculus," by M. Baxter and A. Rennie (Cambridge, 1996). Take a close look at Figure 4.3. Give your opinion on the connection between the figure and the preceding text, specifically on whether the scheme illustrated in the figure appears to converge to the Brownian motion.

doesn't *look like* figure 3.2. At an intuitive level, the global structure of the stock index is different. It grows, gets 'noisier' as time passes, and doesn't go negative. Brownian motion can't be the whole story.

But we only want a basis – the single binomial branching didn't look promising right away. We shouldn't run ahead of ourselves. Brownian motion will prove a remarkably effective component to build continuous processes with – *locally* Brownian motion looks realistic. We should study it closely before we rush on.

Brownian motion

It was nearly a century after botanist Robert Brown first observed microscopic particles zigzagging under the continuous buffeting of a gas that the mathematical model for their movements was properly developed. The first step to the analysis of Brownian motion is to construct a special family of discrete binomial processes.

The random walk $W_n(t)$

For n a positive integer, define the binomial process $W_n(t)$ to have:

(i) $W_n(0) = 0$,

(ii) layer spacing $1/n$,

(iii) up and down jumps equal and of size $1/\sqrt{n}$,

(iv) measure $\mathbb{P}$, given by up and down probabilities everywhere equal to $\frac{1}{2}$.

In other words, if $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the value of W_n at the ith step is defined by:

$$W_n\left(\tfrac{i}{n}\right) = W_n\left(\tfrac{i-1}{n}\right) + \frac{X_i}{\sqrt{n}}, \qquad \text{for all } i \geqslant 1.$$

The first two steps are shown in figure 3.3. What does W_n look like as n gets large?

Instead of blowing out of control, the family portraits (figure 3.4) appear to be settling down towards something as n increases. The moves of size

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$1/\sqrt{n}$ seem to force some kind of convergence. Can we make a formal statement? Consider for example, the distribution of W_n at time 1: for a particular W_n, there are $n+1$ possible values that it can take, ranging from $-\sqrt{n}$ to $\sqrt{n}$. But the distribution always has zero mean and unit variance. (Because $W_n(1)$ is the sum of n IID random variables, each with zero mean and variance $1/n$.)

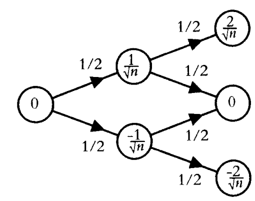

Figure 3.3 The first two steps of the random walk W_n

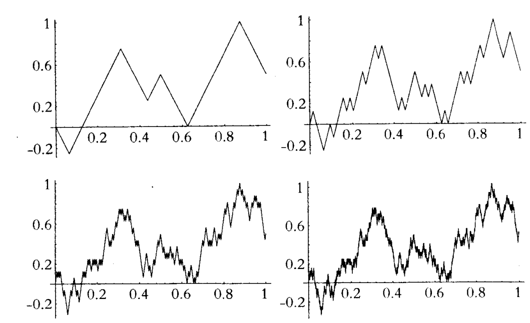

Figure 3.4 Random walks of 16, 64, 256 and 1024 steps respectively

Moreover the central limit theorem gives us a limit for these binomial distributions – as n gets large, the distribution of $W_n(1)$ tends towards the

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2. For the standard Brownian motion B, compute the following.
(a) $P(B(1) + 2B(3) > 1)$.
(b) $P(B(1) + 3B(3) > 5B(5) + 7B(7))$.
(c) $P(B(t) = B(t+1)$ for some *integer* $t \geq 0)$.
(d) $P(B(t) = B(t+1)$ for some *real* $t \geq 0)$.

3. Let $\xi_1, \xi_2, \ldots$ be i.i.d., with $P(\xi_n = \pm 1) = 1/2$, so that $S_n = \xi_1 + \cdots + \xi_n$ is a simple symmetric random walk, and let

$$Z_n = \sum_{k=1}^{n} |S_k|.$$

Find a sequence a_n so that Z_n/a_n converges in distribution to a nontrivial limit X. (*Hint*: X should be a functional of Brownian motion.) Compute $E(X)$ and prove that $E(Z_n)/a_n$ converges to $E(X)$.

Optional. Show that the same statement holds if ξ_n are any i.i.d. random variables with $E(\xi_1) = 0$, $E(\xi_1^2) = 1$.