Math 236A, Fall 2022.

Homework Assignment 4

Due: Nov. 2, 2022

1. Show that the following processes are martingales, and compute their variances.
(a) $X_t = \int_0^t |B_s| \, dB_s$.
(b) $X_t = \int_0^t M_s \, dB_s$, where $M_s = \max\{B_u : 0 \le u \le s\}$.
(c) $X_t = \int_0^t B_{s/2} \, dB_s$.
(d) $X_t = \int_0^t I_s \, dB_s$, where $I_s = \int_0^s B_u^2 \, du$.
(e) $X_t = \int_0^t B_s \bar{B}_s \, dB_s$, where $\bar{B}_s$ is a Brownian motion independent of B_s.

2. Consider $\int_0^t B_{s+1} \, dB_s$, $0 \le t \le 1$.
(a) This is not an Itô integral. Why?
(b) Still, one may hope that, as $n \to \infty$, the limit of the approximating sums $\sum_{i=0}^{n-1} B_{t_i+1}(B_{t_{i+1}} - B_{t_i})$ exists in an appropriate sense. Show that the limit indeed exists (state in what sense), express it with a genuine Itô integral, and show that it results in a continuous process which is *not* a martingale.

3. Assume that $f \in \mathcal{H}^2[0,T]$ and τ is a stopping time with $P(\tau \le T) = 1$. Let X_t be the continuous martingale given by the Itô integral: $X_t = \int_0^t f(s) \, dB(s)$. Show that

$$X_\tau = \int_0^T 1_{[0,\tau]}(s) f(s) \, dB(s)$$

(*Hints*. Call the two sides $I_\ell = I_\ell(f)$ and $I_r = I_r(f)$. You may follow these steps: both I_ℓ and I_r are linear in f; the L^2-norms of $I_\ell(f)$ and $I_r(f)$ are both bounded by a constant times the $\mathcal{H}^2$-norm of f; the claim is true for $f = a \cdot 1_{(u,v]}$, where a is $\mathcal{F}_u$-measurable and $Ea^2 < \infty$.)

Remark. The random variable I_ℓ is defined for *any* random time $\tau \le T$, while I_r is only defined when τ is a stopping time. Indeed, only then is $1_{[0,\tau(\omega)]}(t)$ an adapted process, as then, for a fixed t, the event that this random variable is 1 is $\{t \le \tau\} = \{\tau < t\}^c \in \mathcal{F}_t$. The solution is given in the next page.

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Homework Assignment 4: Solution to problem 3

3. For I_ℓ, linearity in f follows by continuity of X_t and linearity of Itô integral. For I_r, just linearity of Itô integral suffices. Next, by Doob's inequality and Itô isometry,

$$E\left(X_\tau^2\right) \le E\left(\max_{t\in[0,T]} X_t^2\right) \le 4E\left(X_T^2\right) = 4||f||^2_{\mathcal{H}^2[0,T]}.$$

Further by Itô isometry,

$$E(I_r^2) = ||1_{[0,\tau]}f||^2_{\mathcal{H}^2[0,T]} \le ||f||^2_{\mathcal{H}^2[0,T]}.$$

For f as in the hint,

$$X_t = a(B(t\wedge v) - B(t\wedge u)).$$

Assuming τ only has finitely many values t_k, we can write

$$\begin{aligned}
1_{[0,\tau]}(s)f(s) &= a1_{(u,v]}(s) - a1_{\{\tau<s\}}1_{(u,v]}(s)\\
&= a1_{(u,v]}(s) - \sum_k a1_{\{\tau=t_k\}}1_{(t_k,T]}(s)1_{(u,v]}(s)\\
&= a1_{(u,v]}(s) - \sum_{k:t_k\le u} a1_{\{\tau=t_k\}}1_{(u,v]}(s) - \sum_{k:t_k\in(u,v]} a1_{\{\tau=t_k\}}1_{(t_k,v]}(s)
\end{aligned}$$

All summands are functions in $\mathcal{H}_0^2$, and therefore,

$$\begin{aligned}
\int_0^T 1_{[0,\tau]}(s)f(s)\,dB(s) &= a(B(v)-B(u))\\
&\qquad - \sum_{k:t_k\le u} a1_{\{\tau=t_k\}}(B(v)-B(u))\\
&\qquad - \sum_{k:t_k\in(u,v]} a1_{\{\tau=t_k\}}(B(v)-B(t_k))\\
&= \sum_{k:t_k\in(u,v]} a1_{\{\tau=t_k\}}(B(t_k)-B(u)) + \sum_{k:t_k>v} a1_{\{\tau=t_k\}}(B(v)-B(u))\\
&= a(B(\tau\wedge v) - B(\tau\wedge u)) = X_\tau,
\end{aligned}$$

as desired. Now let τ be arbitrary and take a decreasing sequence τ_n of stopping times, with finitely many values, that converge to τ. Then

$$X_{\tau_n} \to X_\tau$$

a.s., by continuity. Moreover

$$1_{[0,\tau_n]}(s)f(s) \to 1_{[0,\tau_n]}(s)f(s)$$

for every ω and s, and therefore by DCT also in $L^2(dP\times dt)$, as

$$|1_{[0,\tau_n]}(s)f(s)| \le |a| \in L^2(dP\times dt).$$

Therefore,

$$X_{\tau_n} = \int_0^T 1_{[0,\tau_n]}(s)f(s)\,dB(s) \to \int_0^T 1_{[0,\tau]}(s)f(s)\,dB(s)$$

in $L^2[0,T]$. This proves that the claim holds for functions f as in the hint, and thus, by linearity, the claim holds for all $f\in\mathcal{H}_0^2[0,T]$. Finally, take an arbitrary $f\in\mathcal{H}_0^2[0,T]$. Pick any sequence $f_n\in\mathcal{H}_0^2[0,T]$ so that $f_n\to f$ in $\mathcal{H}^2[0,T]$, and use the two norm estimates to conclude that both $I_\ell(f_n)\to I_\ell(f)$ and $I_r(f_n)\to I_r(f)$ in $L^2(dP)$. As $I_\ell(f_n) = I_r(f_n)$ for all n, $I_\ell(f) = I_r(f)$.