Math 236A, Fall 2022.

Homework 5

Due: Nov. 9, 2022

Note. These problems are on applications of Itô formula.

1. Compute the following stochastic integrals. The answers may contain Riemann integrals.
(a) $\int_0^t e^s \, dB_s$.

(b) $\int_0^t B_s^2 \, dB_s$.

(c) $\int_0^t s B_s \, dB_s$.

2. Fix $a > 0$ and let τ be the first time that Brownian motion B_t hits either a or $-a$. Compute, with careful proof:

$$E \int_0^\tau B_s^2 \, ds.$$

3. (a) Assume that $f = f(t, x)$ is a $\mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ such that $\partial_t f + \frac{1}{2}\partial_{xx} f = 0$. Show that $f(t, B_t)$ is a local martingale.
(b) Let h_n, $n = 0, 1, \ldots$, be the Hermite polynomial of degree n. (The first four are 1, x, $x^2 - 1$, $x^3 - 3x$.) Let $f_n(t, x) = t^{n/2} h_n(x/\sqrt{t})$ for $n \geq 0$. Show that $f_n(t, B_t)$ is a martingale for every n.
(c) Define the processes X_t^n recursively by $X_t^0 \equiv 1$ and $X_t^{n+1} = \int_0^t X_s^n \, dB_s$. Find the connection between X_t^n and the martingales from (b).
Note. You will need three facts about Hermite polynomials: they are eigenvectors of the Hermite equation,

$$h_n'' - x h_n' = -n h_n;$$

that each of their derivatives is a multiple of the lower-degree Hermite polynomial,

$$h_n' = n h_{n-1};$$

and the second order recurrence relation

$$h_{n+1} = x h_n - n h_{n-1}.$$