Final Exam

NAME (print in CAPITAL letters, first name first): __________________________

NAME (sign): ________________________________________

ID#: ______________________________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 10 pages (including this one) with 8 problems.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
</tr>
</tbody>
</table>
1. Assume \( x_1 = a \) and
\[
x_{n+1} = \frac{2}{5 - 2x_n} \quad \text{for } n \in \mathbb{N}.
\]

(a) Assume \( a = 1 \). Show that \( 0 \leq x_n \leq 2 \) for all \( n \in \mathbb{N} \), and that the sequence is decreasing. Then show that \( \lim_{n \to \infty} x_n \) exists and compute the limit.

To prove \( 0 \leq x_n \leq 2 \) for all \( n \in \mathbb{N} \):

\( (n = 1) \) \( x_1 = 1 \in [0, 2] \).

\( (n \to n+1) \) Assume \( x_n \in [0, 2] \), then \( 5 - 2x_n \in [1, 5] \) and \( \frac{2}{5 - 2x_n} \in \left[ \frac{2}{5}, 2 \right] \subseteq [0, 2] \).

To prove that \( x_{n+1} < x_n \) for all \( n \in \mathbb{N} \):

\( (n = 1) \) \( x_2 = \frac{2}{3} < 1 = x_1 \).

\( (n \to n+1) \) If \( x_{n+1} < x_n \), then \( 5 - 2x_{n+1} > 5 - 2x_n \) and (as we know that both sides are positive) \( x_{n+1} \leq 2 \) implies \( \frac{2}{5 - 2x_{n+1}} \leq \frac{2}{5 - 2x_n} \), that is, \( x_{n+2} < x_{n+1} \).

Therefore, \( x_n \) is a decreasing bounded sequence, thus convergent. Let \( x = \lim x_n \), then
\[
x = \frac{2}{5 - 2x}, \quad 5x - 2x^2 = 2;
\]
\[
2x^2 - 5x + 2 = 0, \quad x = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm 3}{4}
\]
\[
x = \frac{1}{2} \text{ or } 2 \text{. But } x_0 < 2 \text{ and the sequence decreases, so } x < 2 \text{.}
\]

Answer. \( \lim_{n \to \infty} x_n = x = \frac{1}{2} \).
Problem 1, continued.

(b) Now assume \( a = 0 \). Show that the sequence is increasing and that it converges to the same limit as in (a).

The proof that \( 0 \leq x_n \leq 2 \) for all \( n \) from (a) is still valid. To prove by induction that \( x_{n+1} > x_n \) for all \( n \) by induction:

\( n = 1 \) \[ x_2 = \frac{2}{5} > x_1, \]

\( n \to n + 1 \) \[ x_{n+1} > x_n \Rightarrow \frac{5 - 2x_{n+1}}{5 - 2x_n} > \frac{1}{5 - 2x_n} \] both > 0

\[ \Rightarrow \frac{1}{5 - 2x_{n+1}} > \frac{1}{5 - 2x_n} \Rightarrow x_{n+2} > x_{n+1} \]

Also, \( x_n \leq \frac{1}{2} \):

\( n = 1 \) \[ x_1 = 0 \leq \frac{1}{2} \]

\( n \to n + 1 \) If \( x_n \leq \frac{1}{2} \), then \( 5 - 2x_n \geq 4 \) and \( \frac{2}{5 - 2x_n} \leq \frac{1}{2} \), so \( x_n \leq \frac{1}{2} \).

The sequence is increasing and bounded by \( \frac{1}{2} \), so \( \lim_{n \to \infty} x_n \) exists and \( x \leq \frac{1}{2} \).

Also, \( x \) must satisfy the same equation as in (a). So \( x = \frac{1}{2} \).
2. (a) Assume \( A \subseteq \mathbb{R} \). Define precisely what this statement means: \( A \) is an open set.

For every \( x \in A \), there exist an \( \varepsilon > 0 \) so that \( (x-\varepsilon, x+\varepsilon) \subseteq A \).

(b) Is the set \([0,1]\) open?

No. \( 0 \in [0,1] \), but for every \( \varepsilon > 0 \), \( (-\varepsilon, \varepsilon) \not\subseteq [0,1] \).

(c) Is the set \( \bigcup_{n=1}^{\infty} (n, n+1/n) \) open?

Yes. Any union of open sets (open intervals in this case) is open.

(d) Determine the boundary and the interior of the set \([0,2] \setminus \{1\}\).

\[ [0,2] \setminus \{1\} = [0,1) \cup (1,2] \]

Interior: \( (0,1) \cup (1,2) \)

Boundary: \( [0,1,2] \)
3. (a) Assume $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Define precisely what these two statements mean: $x$ is a limit point of $A$; $A$ is a closed set.

$x$ is a limit point: for every $\varepsilon > 0$,

$A \cap (V_{\varepsilon}(x) \setminus \{x\}) \neq \emptyset$.

$A$ is closed: $A$ contains all its limit points.

(We denote $V_{\varepsilon}(x) = (x-\varepsilon, x+\varepsilon)$.)

(b) Assume $A$ is bounded above. Assume that $\sup A \notin A$. Show that $\sup A$ is a limit point of $A$.

Let $s = \sup A$ and $\varepsilon > 0$. Then there exists an $a \in A$ so that $a > s - \varepsilon$. As $s$ is an upper bd. for $A$ but $s \notin A$, $a < s$, therefore $a \in (s-\varepsilon, s)$ and $a \in (V_{\varepsilon}(s) \setminus \{s\}) \cap A \neq \emptyset$.

(c) Assume $A$ is bounded above and closed. Prove that $\sup A \in A$.

Assume $\sup A \notin A$.

We proved in (b) that $\sup A$ is then a limit pt. of $A$, so $A$ does not contain one of its limit pts; thus $A$ is not closed.

(d) Assume $A$ is bounded above and open. Prove that $\sup A \notin A$.

Let $s = \sup A$.

If $s \in A$, and $A$ is open, there is an $\varepsilon > 0$ so that $(s-\varepsilon, s+\varepsilon) \subseteq A$. But this means, for example that $s < s + \varepsilon / 2 \in A$ and $s$ is not an upper bound for $A$, contradiction.
4. (a) Assume $A \subseteq \mathbb{R}$. Define precisely what this statement means: $A$ is compact.

Every sequence $(x_n)$ with $x_n \in A$ has a subsequence $(x_{n_k})$ with $\lim_{k \to \infty} x_{n_k} \in A$.

(b) Is the set $\{\frac{n}{n+1} : n \in \mathbb{N}\}$ compact?

No, $x_n = \frac{n}{n+1}$ is not an element on this set converges to 0 which is not in the set. Any subsequence $(x_{n_k})$ then also converges to 0.

(c) Is the set $[0, 1] \cup \{2\}$ compact?

Yes. By Heine–Borel Theorem, any closed and bounded set is compact. This set is obviously bounded and is closed as a union of two closed sets $[0, 1]$ and $\{2\}$.

(d) True or false: If $A^c$ is compact, then $A$ is open and unbounded.

True. $A^c$ is compact $\Rightarrow A^c$ closed $\Rightarrow A$ open.

Also, $A$ cannot be bounded. If $A^c$ is bounded (as a compact set), to if $A$ were also bounded, then $A \cup A^c = \mathbb{R}$ would also be bounded, contradiction.

(e) Assume $A$ is compact and $\epsilon > 0$. Prove that there exist finitely many elements $x_1, \ldots, x_n \in A$ so that $A \subseteq \bigcup_{i=1}^{n} (x_i - \epsilon, x_i + \epsilon)$.

Take the open cover $\mathcal{F} = \{ (x_i - \epsilon, x_i + \epsilon) : x_i \in A \}$. Then there exists a finite subcover $\mathcal{F}' = \{ (x_1 - \epsilon, x_1 + \epsilon), \ldots, (x_n - \epsilon, x_n + \epsilon) \}$, for some $x_1, \ldots, x_n \in A$. This means $A \subseteq \bigcup_{i=1}^{n} (x_i - \epsilon, x_i + \epsilon)$.
5. For each of the following series, determine (with proof) whether it converges absolutely, converges conditionally, or diverges:

(a) \[ \sum_{k=1}^{\infty} \frac{(7k+3)^k}{8k+5} \]

Converges absolutely:

Root test

\[ \lim_{k \to \infty} a_k^{1/k} = \left( \frac{7k+3}{8k+5} \right)^{7/8} < 1. \]

(b) \[ \sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k^2+1} \]

Converges absolutely

|a_k| = \frac{\sqrt{k}}{k^2+1} \leq \frac{1}{k^{3/2}}

and \( \sum \frac{1}{k^{3/2}} \) converges as \( 3/2 > 1 \).

(c) \[ \sum_{k=1}^{\infty} \frac{(-1)^k}{3k-2} \]

Converges conditionally:

As \( \frac{1}{3k-2} \) decreases to 0, the term converges by the alternating series test. However, \( |a_k| = \frac{1}{3k-2} \) and \( \lim_{k \to \infty} \frac{k}{3k-2} = \frac{1}{3} \). The term \( \sum |a_k| \) diverges by the limit comparison test.

(d) \[ \sum_{k=1}^{\infty} \left( \frac{1}{k} + \frac{1}{2k} \right) \]

Diverges

\[ \frac{1}{k} + \frac{1}{2k} \geq \frac{1}{k} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} = \infty \] (Harmonic series)
6. Assume that \( a_n > 0 \) for all \( n \in \mathbb{N} \). For each statement below, prove it or find a counterexample.

(a) If \( \sum_{n=1}^{\infty} (a_n - 5) \) converges, then \( \lim a_n = 5 \).

\[
\text{Yes. } \lim_{n \to \infty} (a_n - 5) = 0, \text{ so } \lim_{n \to \infty} a_n = 5.
\]

(b) If \( a_{n+1} < a_n \) for all \( n \in \mathbb{N} \), then the sequence \( (a_n) \) is Cauchy.

\[
\text{Yes. Such sequences are bounded (all terms are in } [0, a_0] \text{) and monotonically increasing, thus convergent, thus Cauchy.}
\]

(c) If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} \sqrt{a_n} \) converges.

\[
\text{No. Take } a_n = \frac{1}{n^2}.
\]

(d) If \( \limsup n^2 a_n = 1 \), \( \sum_{n=1}^{\infty} a_n \) converges.

\[
\text{Yes. Except for finitely many } n, \text{ } n^2 a_n \leq 2, \text{ thus: } a_n \leq \frac{2}{n^2}. \text{ The series converges by comparison with } \sum \frac{2}{n^2}, \text{ a convergent series.}
\]
7. (a) Assume \((a_n)\) is a sequence of real numbers and \(a \in \mathbb{R}\). Define precisely what this statement means: \(\lim a_n = a\).

For every \(\varepsilon > 0\), there is an \(N \in \mathbb{N}\), so that \(|a_n - a| < \varepsilon\) for \(n \geq N\).

(b) Assume \(\lim a_n = a\). Explain why this statement is false: There exists an \(\varepsilon > 0\) so that there are infinitely many terms of \((a_n)\) outside \((a - \varepsilon, a + \varepsilon)\).

This statement is false because it is the exact negation of the one in (a).

(c) Let \(a_n = \frac{n^2 + n}{n^2 + 1}\). Compute \(a = \lim a_n\). (You may use algebraic and order limit theorems, but give full justification.)

\[
a_n = \frac{1 + \frac{1}{n}}{1 + \frac{1}{n^2}}
\]

As \(\frac{1}{n} \to 0\) and \(\frac{1}{n^2} \to 0\),

\[
\lim a_n = 1.
\]

(d) Let \(a_n = \frac{3^n + 2^n}{3^{n+1} + 2^{n+1}}\). Compute \(a = \lim a_n\). (Again, you may use algebraic and order limit theorems.)

\[
a_n = \frac{1 + \left(\frac{2}{3}\right)^n}{3 + 2\left(\frac{2}{3}\right)^n}
\]

As \(\left(\frac{2}{3}\right)^n \to 0\) \((\frac{2}{3} < 1)\),

\[
\lim a_n = \frac{1}{3}.
\]
8. Assume \((a_n)\) is a sequence of real numbers.
(a) True or false: If the sequence is bounded, then it has a convergent subsequence.

**True.** This is Bolzano-Weierstrass theorem.

(b) True or false: If the sequence is unbounded, then it has no convergent subsequence.

**Not true.** Take the sequence \((a_n)\) given by 
\[ a_n = \begin{cases} 0 & n \text{ even} \\ n & n \text{ odd} \end{cases} \]
Then \((a_n)\) is unbounded.

by \(a_{2k} = 0\) is the convergent subsequence.

(c) Prove: If the sequence is unbounded, and \(a_n \geq 0\) for all \(n \in \mathbb{N}\), then there exists a subsequence that diverges to \(\infty\). Because \((a_n)\) is unbounded, for any \(k \in \mathbb{N}\), we can choose \(n_k\) so that \(a_{n_k} \geq k\) and so that \(n_k > n_{k-1}\).

Then \(\lim a_{n_k} \geq \lim k = \infty\).

(d) Assume that \(\lim(4a_n - a_n^2) = 3\), and that \(\lim a_n\) does not exist. Determine \(\lim sup a_n\) and \(\lim inf a_n\).

As \(4a_n - a_n^2 = a_n(4-a_n) \geq 0\) except for finitely many \(n\), \(0 \leq a_n \leq 4\) except for finitely many \(n\) (if either \(a_n < 0\) or \(a_n > 4\), \(a_n(4-a_n) < 0\)). Thus any convergent subsequence has finite limit. If \(x\) is any subsequential limit, then \(4x - x^2 = 3\) and \(x^2 - 4x + 3 = 0\), \((x-3)(x-1) = 0\) \(x = 1\) or \(x = 3\). Since \(\lim a_n\) does not exist, both \(1\) and \(3\) must be subsequential limits. So \(\lim sup a_n = 3\) and \(\lim inf a_n = 1\).